Convex Optimization

(EE227A: UC Berkeley)

Lecture 8

Weak duality

14 Feb, 2013

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Primal problem

Let $f_i : \mathbb{R}^n \to \mathbb{R}$ $(0 \leq i \leq m)$. Generic **nonlinear program**

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \quad 1 \leq i \leq m, \\ & x \in \{\operatorname{dom} f_0 \cap \operatorname{dom} f_1 \cdots \cap \operatorname{dom} f_m\}. \end{aligned} \tag{P}$$

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- In our initial derivation: no restriction to convexity.

Lagrangian

To the primal problem, associate **Lagrangian** $\mathcal{L} : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$,

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- ♠ Variables $\lambda \in \mathbb{R}^m$ called **Lagrange multipliers**
- ♠ Suppose x is feasible, and $\lambda \geq 0$. Then, we get the lower-bound:

$$f_0(x) \geq \mathcal{L}(x, \lambda) \qquad \forall x \in \mathcal{X}, \quad \lambda \in \mathbb{R}^m_+.$$

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- Lagrangian helps write problem in **unconstrained form**

Lagrangian

Claim: Since, $f_0(x) \geq \mathcal{L}(x, \lambda) \quad \forall x \in \mathcal{X}, \quad \lambda \in \mathbb{R}_+^m$, primal optimal

$$p^* = \inf_{x \in \mathcal{X}} \sup_{\lambda \geq 0} \quad \mathcal{L}(x, \lambda).$$

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- ♠ In this case, inner $\sup$ is $+\infty$, so claim true by definition
- ♠ If x is feasible, each $f_i(x) \leq 0$, so $\sup_\lambda \sum_i \lambda_i f_i(x) = 0$

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- $\forall x \in \mathcal{X}, \quad f_0(x) \geq \inf_{x'} \mathcal{L}(x', \lambda) = g(\lambda)$
- Now minimize over x on lhs, to obtain

$$\forall\ \lambda \in \mathbb{R}^m_+ \qquad p^* \geq g(\lambda).$$

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- **dual optimal:** λ^* if $\sup$ is achieved
- Lagrange dual is always concave, regardless of original

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Theorem (Weak-duality): For problem (P), we have $p^* \geq d^*$.

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Theorem (Weak-duality): For problem (P), we have $p^* \geq d^*$.

Proof: We showed that for all $\lambda \in \mathbb{R}^m_+$, $p^* \geq g(\lambda)$.
Thus, it follows that $p^* \geq \sup g(\lambda) = d^*$.

Equality constraints

$$\begin{aligned} \min \quad & f_0(x) \\ \text{s.t.} \quad & f_i(x) \leq 0, \quad i = 1, \ldots, m, \\ & h_i(x) = 0, \quad i = 1, \ldots, p. \end{aligned}$$

Exercise: Show that we get the Lagrangian dual

$$g : \mathbb{R}^m_+ \times \mathbb{R}^p : (\lambda, \nu) \mapsto \inf_x \quad \mathcal{L}(x, \lambda, \nu),$$

where the Lagrange variable ν corresponding to the equality constraints is unconstrained.
Hint: Represent $h_i(x) = 0$ as $h_i(x) \leq 0$ and $-h_i(x) \leq 0$.

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Again, we see that $p^* \geq \sup_{\lambda \geq 0, \nu} \; g(\lambda, \nu) = d^*$

Some duals

- Least-norm solution of linear equations: $\min x^T x$ s.t. $Ax = b$
- Linear programming standard form
- Study example (5.7) in BV (binary QP)