Convex Optimization

(EE227A: UC Berkeley)

Lecture 4
(Conjugates, subdifferentials)

31 Jan, 2013

Suvrit Sra

Organizational

- ♡ HW1 due: **14th Feb 2013** in class.
- ♡ Please LaTeX your solutions (contact TA if this is an issue)
- ♡ Discussion with classmates is ok
- ♡ Each person must submit his/her individual solutions
- ♡ Acknowledge any help you receive
- ♡ Do not copy!
- ♡ Make sure you understand the solution you submit
- ♡ Cite any source that you use
- ♡ Have fun solving problems!

Recap

- Eigenvalues, singular values, positive definiteness
- Convex sets, $\theta_1 x + \theta_2 y \in C$, $\theta_1 + \theta_2 = 1$, $\theta_i \geq 0$
- Convex functions, midpoint convex, recognizing convexity
- Norms, mixed-norms, matrix norms, dual norms
- Indicator, distance function, minimum of jointly convex
- Brief mention of other forms of convexity

$$f\left(\frac{x+y}{2}\right) \leq \frac{1}{2}[f(x) + f(y)] + \text{continuity} \implies f \text{ is cvx}$$

$$\nabla^2 f(x) \succeq 0 \text{ implies } f \text{ is cvx.}$$

Fenchel Conjugate

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Hint: Use *Hölder's inequality:* $u^T v \leq \|u\|_p \|v\|_q$

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- Thus, $f(z) = +\infty$ if (i), and 0 if (ii), as desired.

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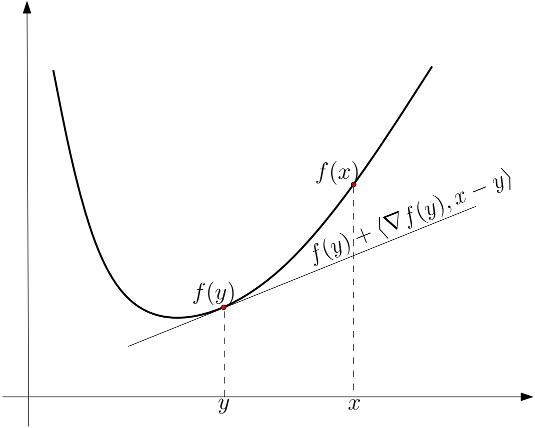
Exercise: If $f(x) = \max(0, 1-x)$, then $\operatorname{dom} f^*$ is $[-1, 0]$, and within this domain, $f^*(z) = z$.

Hint: Analyze cases: $\max(0, 1-x) = 0$; and $\max(0, 1-x) = 1-x$

Subdifferentials

First order global underestimator



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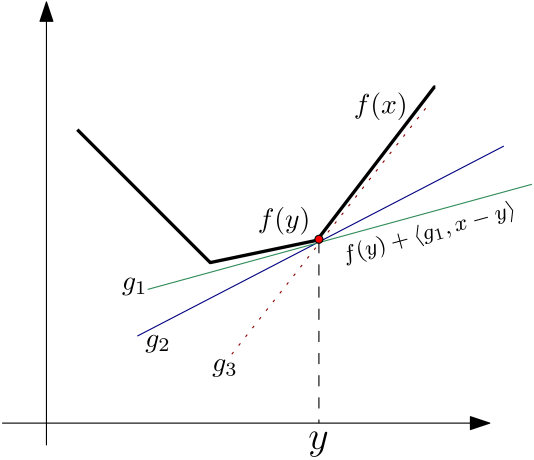

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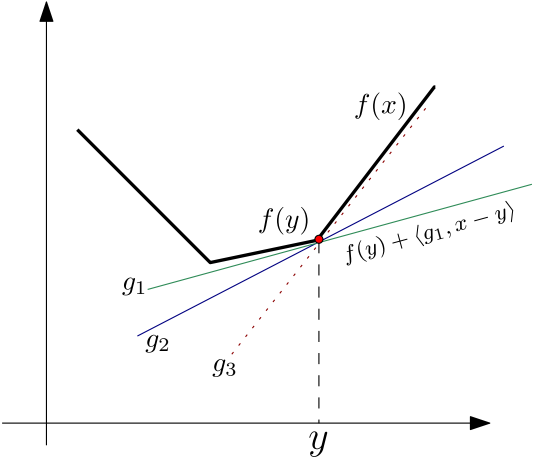
Subgradients



g_1, g_2, g_3 are subgradients at y

Subgradients – basic facts

- f is convex, differentiable: $\nabla f(y)$ the **unique** subgradient at y
- A vector g is a subgradient at a point y if and only if $f(y) + \langle g, x - y \rangle$ is **globally** smaller than $f(x)$.
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- Without convexity, things become wild! — advanced course

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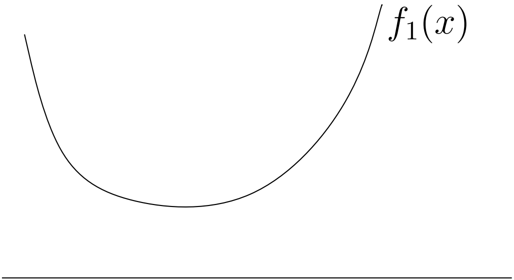

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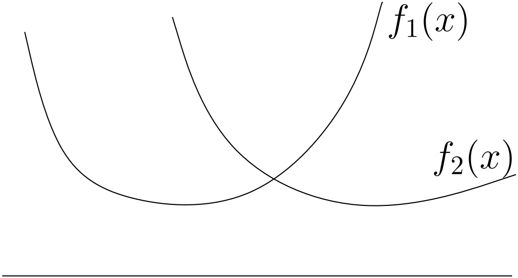

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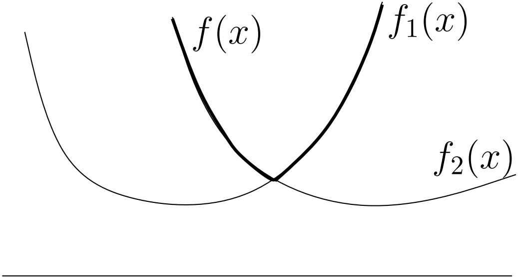

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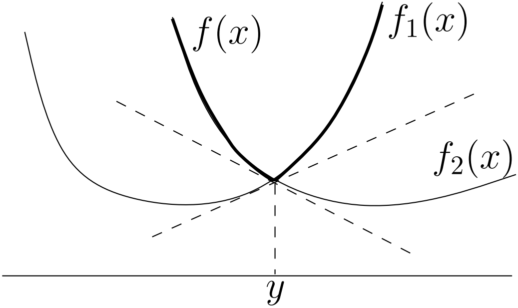

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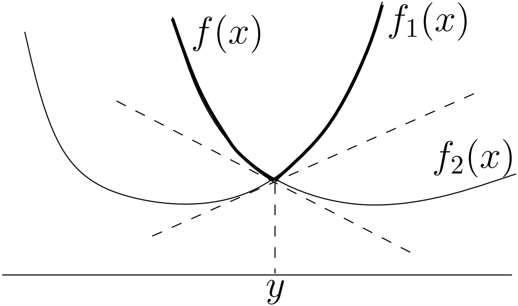

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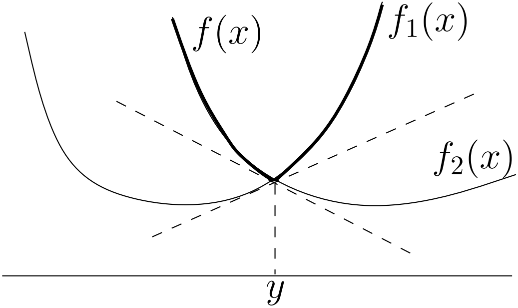

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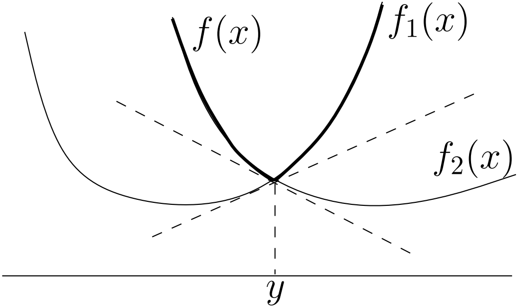

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- ⋆ $f_1(y) = f_2(y)$: subgradients, the segment $[f_1'(y), f_2'(y)]$ (imagine all supporting lines turning about point y)

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- ♣ If $\partial f(x) = \{g\}$, then f is differentiable and $g = \nabla f(x)$

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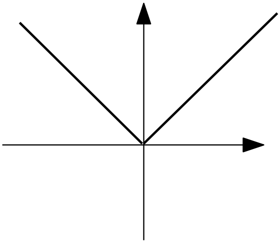

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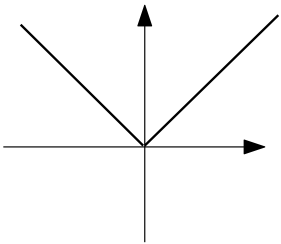

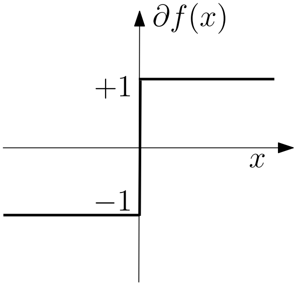

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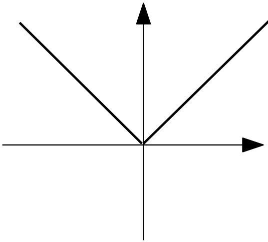

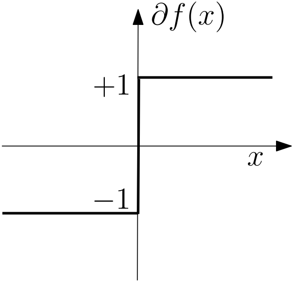

$$\partial|x| = \begin{cases} -1 & x < 0, \\ +1 & x > 0, \\ [-1, 1] & x = 0. \end{cases}$$

More examples

Example $f(x) = \|x\|_2$. Then,

$$\partial f(x) := \begin{cases} \|x\|_2^{-1} x & x \neq 0, \\ \{z \mid \|z\|_2 \leq 1\} & x = 0. \end{cases}$$

More examples

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More examples

Example A convex function need not be subdifferentiable everywhere.

Let

$$f(x) := \begin{cases} -(1 - \|x\|_2^2)^{1/2} & \text{if } \|x\|_2 \le 1, \\ +\infty & \text{otherwise.} \end{cases}$$

f diff. for all x with $\|x\|_2 < 1$, but $\partial f(x) = \emptyset$ whenever $\|x\|_2 \ge 1$.

Calculus

Recall basic calculus

If f and k are differentiable, we know that

- **Addition**: $\nabla(f+k)(x) = \nabla f(x) + \nabla k(x)$
- **Scaling**: $\nabla(\alpha f(x)) = \alpha \nabla f(x)$

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Chain rule

If $f : \mathbb{R}^n \to \mathbb{R}^m$, and $k : \mathbb{R}^m \to \mathbb{R}^p$. Let $h : \mathbb{R}^n \to \mathbb{R}^p$ be the composition $h(x) = (k \circ f)(x) = k(f(x))$. Then,

$$Dh(x) = Dk(f(x))Df(x).$$

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Example If $f : \mathbb{R}^n \to \mathbb{R}$ and $k : \mathbb{R} \to \mathbb{R}$, then using the fact that $\nabla h(x) = [Dh(x)]^T$, we obtain

$$\nabla h(x) = k'(f(x))\nabla f(x).$$

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$$\partial f(x) = \text{conv} \bigcup \{\partial f_i(x) \mid f_i(x) = f(x)\},$$

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convex hull over subdifferentials of "active" functions at x
- **Conjugation:** $z \in \partial f(x)$ if and only if $x \in \partial f^*(z)$

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But $\partial f_1(0) = \partial f_2(0) = \emptyset$.

However, $\partial f_1(x) + \partial f_2(x) \subset \partial(f_1 + f_2)(x)$ always holds.

Examples

Example $f(x) = \|x\|_\infty$. Then,

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where e_i is i-th canonical basis vector (column of identity matrix).

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To prove, notice that $f(x) = \max_{1 \le i \le n} \left\{|e_i^T x|\right\}$.

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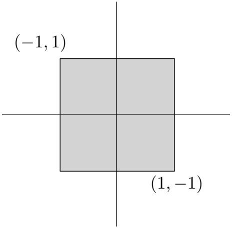

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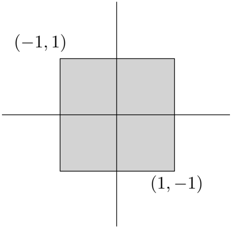

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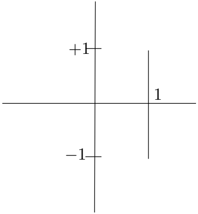

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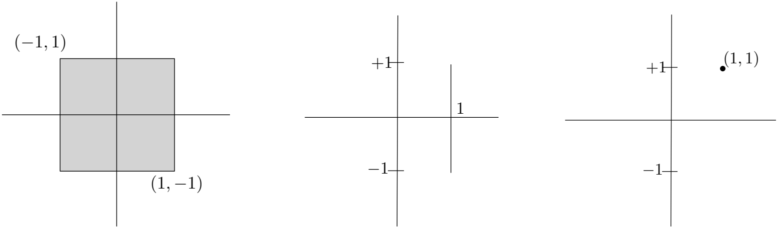

∂f at $x = (0, 0)$ ∂f at $x = (1, 0)$ ∂f at $x = (1, 1)$

Rules for subgradients

Subgradient for pointwise sup

$$f(x) := \sup_{y \in \mathcal{Y}} \quad h(x, y)$$

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h(z, y^*) &\geq h(x, y^*) + g^T(z - x) \\
h(z, y^*) &\geq f(x) + g^T(z - x) \\
f(z) &\geq h(z, y) \qquad \text{(because of sup)} \\
f(z) &\geq f(x) + g^T(z - x).
\end{aligned}$$

Example

Suppose $a_i \in \mathbb{R}^n$ and $b_i \in \mathbb{R}$. And

$$f(x) := \max_{1 \leq i \leq n} (a_i^T x + b_i).$$

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- Here $f(x; y) = f_k(x) = a_k^T x + b_k$, and $\partial f_k(x) = \{\nabla f_k(x)\}$
- Hence, $a_k \in \partial f(x)$ works!

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- Then, $g = \int g(x, u) p(u) du = \mathbf{E} g(x, u) \in \partial f(x)$

Subgradient of composition

Suppose $h : \mathbb{R}^n \to \mathbb{R}$ cvx and nondecreasing; each f_i cvx

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Exercise: Verify $g \in \partial f(x)$ by showing $f(z) \geq f(x) + g^T(z - x)$

References

1. R. T. Rockafellar. *Convex Analysis*
2. S. Boyd (Stanford); EE364b Lecture Notes.