On inequalities for normalized Schur functions

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A R T I C L E   I N F O

Article history:
Received 16 February 2015
Accepted 17 July 2015

A B S T R A C T

We prove a conjecture of Cuttler et al. (2011) on the monotonicity of normalized Schur functions under the usual (dominance) partial-order on partitions. We believe that our proof technique may be helpful in obtaining similar inequalities for other symmetric functions.

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We prove a conjecture of Cuttler et al. [1] on the monotonicity of normalized Schur functions under the majorization (dominance) partial-order on integer partitions.

Schur functions are one of the most important bases for the algebra of symmetric functions. Let \( \mathbf{x} = (x_1, \ldots, x_n) \) be a tuple of \( n \) real variables. Schur functions of \( \mathbf{x} \) are indexed by integer partitions \( \lambda = (\lambda_1, \ldots, \lambda_n) \), where \( \lambda_1 \geq \cdots \geq \lambda_n \), and can be written as the following ratio of determinants [7, pg. 49], [5, (3.1)]:

\[
s_{\lambda}(\mathbf{x}) = s_{\lambda}(x_1, \ldots, x_n) := \frac{\det([x_i^{\lambda_j+n-j}])_{i,j=1}^n}{\det([x_i^{n-j}])_{i,j=1}^n}.
\]  

(0.1)

To each Schur function \( s_{\lambda}(\mathbf{x}) \) we can associate the normalized Schur function

\[
S_{\lambda}(\mathbf{x}) \equiv S_{\lambda}(x_1, \ldots, x_n) := \frac{s_{\lambda}(x_1, \ldots, x_n)}{s_{\lambda}(1, \ldots, 1)} = \frac{s_{\lambda}(\mathbf{x})}{s_{\lambda}(1^n)}.
\]  

(0.2)

Let \( \lambda, \mu \in \mathbb{R}^n \) be decreasingly ordered. We say \( \lambda \) is majorized by \( \mu \), denoted \( \lambda < \mu \), if

\[
\sum_{i=1}^k \lambda_i \leq \sum_{i=1}^k \mu_i \quad \text{for } 1 \leq i \leq n-1, \quad \text{and} \quad \sum_{i=1}^n \lambda_i = \sum_{i=1}^n \mu_i.
\]  

(0.3)

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http://dx.doi.org/10.1016/j.ejc.2015.07.005
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Cutler et al. [1] studied normalized Schur functions (0.2) among other symmetric functions, and derived inequalities for them under the partial-order (0.3). They also conjectured related inequalities, of which perhaps Conjecture 1 is the most important.

**Conjecture 1** ([1]). Let $\lambda$ and $\mu$ be partitions; and let $x \geq 0$. Then,

$$S_{\lambda}(x) \leq S_{\mu}(x), \quad \text{if and only if} \quad \lambda \prec \mu.$$  

Cutler et al. [1] established necessity (i.e., $S_{\lambda} \leq S_{\mu}$ only if $\lambda \prec \mu$), but sufficiency was left open. We prove sufficiency in this paper.

**Theorem 2.** Let $\lambda$ and $\mu$ be partitions such that $\lambda \prec \mu$, and let $x \geq 0$. Then,

$$S_{\lambda}(x) \leq S_{\mu}(x).$$

Our proof technique differs completely from [1]: instead of taking a direct algebraic approach, we invoke a well-known integral from random matrix theory. We believe that our approach might extend to yield inequalities for other symmetric polynomials such as Jack polynomials [4] or even Hall–Littlewood and Macdonald polynomials [5].

## 1. Majorization inequality for Schur polynomials

Our main idea is to represent normalized Schur polynomials (0.2) using an integral compatible with the partial-order `$\prec$'. One such integral is the Harish-Chandra–Itzykson–Zuber (HCIZ) integral [2,3]:

$$I(A, B) := \int_{U(n)} e^{\text{tr}(U^*AU)} dU = c_n \frac{\det([e^{a_i b_j}])_{i,j=1}^n}{\Delta(a) \Delta(b)},$$

(1.1)

where $dU$ is the Haar probability measure on the unitary group $U(n)$; $a$ and $b$ are vectors of eigenvalues of the Hermitian matrices $A$ and $B$; $\Delta$ is the Vandermonde determinant $\Delta(a) := \prod_{1 \leq i < j \leq n} (a_j - a_i)$; and $c_n$ is the constant

$$c_n = \left(\prod_{i=1}^{n-1} i!\right) = \Delta([1, \ldots, n]) = \prod_{1 \leq i < j \leq n} (j - i).$$

(1.2)

The following observation [2] is of central importance to us.

**Proposition 3.** Let $A$ be a Hermitian matrix, $\lambda$ an integer partition, and $B$ the diagonal matrix $\text{Diag}([\lambda_j + n - j])_{i=1}^n$. Then,

$$\frac{s_\lambda(e^{a_1}, \ldots, e^{a_n})}{s_\lambda(1, \ldots, 1)} = \frac{1}{E(A)} I(A, B),$$

(1.3)

where the product $E(A)$ is given by

$$E(A) = \prod_{1 \leq i < j \leq n} \frac{e^{a_i} - e^{a_j}}{a_i - a_j}.$$  

(1.4)

**Proof.** Recall from Weyl’s dimension formula that

$$s_\lambda(1, \ldots, 1) = \prod_{1 \leq i < j \leq n} \frac{(\lambda_i - i) - (\lambda_j - j)}{j - i}.$$  

(1.5)

Now use identity (1.5), definition (1.2), and the ratio (0.1) in (1.1), to obtain (1.3). □

Assume without loss of generality that for each $i$, $x_i > 0$ (for $x_i = 0$, apply the usual continuity argument). Then, there exist reals $a_1, \ldots, a_n$ such that $e^{a_i} = x_i$, whereby

$$S_\lambda(x_1, \ldots, x_n) = \frac{s_\lambda(e^{\log x_1}, \ldots, e^{\log x_n})}{s_\lambda(1, \ldots, 1)} = \frac{I(\log X, B(\lambda))}{E(\log X)},$$

(1.6)
where $X = \text{Diag}([x_i]_{i=1}^n)$; we write $B(\lambda)$ to explicitly indicate $B$’s dependence on $\lambda$ as in Proposition 3. Since $E(\log X) > 0$, to prove Theorem 2, it suffices to prove Theorem 4 instead.

**Theorem 4.** Let $X$ be an arbitrary Hermitian matrix. Define the map $F : \mathbb{R}^n \to \mathbb{R}$ by

$$F(\lambda) := I(X, \text{Diag}(\lambda)), \quad \lambda \in \mathbb{R}^n.$$ 

Then, $F$ is Schur-convex, i.e., if $\lambda, \mu \in \mathbb{R}^n$ such that $\lambda < \mu$, then $F(\lambda) \leq F(\mu)$.

**Proof.** We know from [6, Proposition C.2, pg. 97] that a convex and symmetric function is Schur-convex. From the HCIZ integral (1.1) symmetry of $F$ is apparent; to establish its convexity it suffices to demonstrate midpoint convexity:

$$F\left(\frac{\lambda + \mu}{2}\right) \leq \frac{1}{2}F(\lambda) + \frac{1}{2}F(\mu) \quad \text{for } \lambda, \mu \in \mathbb{R}^n. \quad (1.7)$$

The elementary manipulations below show that inequality (1.7) holds.

$$F\left(\frac{\lambda + \mu}{2}\right) = \int_{U(n)} \exp(\text{tr}[U^*XU \text{Diag}(\frac{\lambda + \mu}{2})])dU$$

$$= \int_{U(n)} \exp\left(\frac{1}{2}\text{tr}[U^*XU \text{Diag}(\lambda)] + \frac{1}{2}\text{tr}[U^*XU \text{Diag}(\mu)]\right) dU$$

$$= \int_{U(n)} \sqrt{\exp(\text{tr}[U^*XU \text{Diag}(\lambda)]) \cdot \exp(\text{tr}[U^*XU \text{Diag}(\mu)])} dU$$

$$\leq \frac{1}{2} \exp\left(\frac{1}{2}\text{tr}[U^*XU \text{Diag}(\lambda)]\right) + \frac{1}{2} \exp\left(\frac{1}{2}\text{tr}[U^*XU \text{Diag}(\mu)]\right)$$

$$= \frac{1}{2}F(\lambda) + \frac{1}{2}F(\mu),$$

where the inequality follows from the arithmetic-mean geometric-mean inequality. \qed

**Corollary 5.** Conjecture 1 is true.

**Acknowledgments**

I am grateful to a referee for uncovering an egregious error in my initial attempt at Theorem 4; thanks also to the same or different referee for the valuable feedback and encouragement. I thank Jonathan Novak (MIT) for his help with HCIZ references.

**References**