Inequalities via symmetric polynomial majorization

(Original: 18 Aug 2015; Update: 18 Sep 2015)

Suvrit Sra suvrit@mit.edu
Laboratory for Information and Decision Systems
Massachusetts Institute of Technology
Cambridge, MA, 02139, United States

Abstract

We consider a partial order on positive vectors induced by elementary symmetric polynomials. As a corollary we obtain a short proof of the SSLI inequality of [6], which was first obtained via a more elaborate approach.[1] Our proofs are based on a simple observation that uses suitable integral representations and yields a family of monotonicity inequalities under a partial order determined by elementary symmetric polynomials, and thereby yields elementary proofs of the inequality of [6] as well as related entropy inequalities of [3].

Acknowledgments

The author thanks L. Borisov, P. Neff, and C. Thiel for their joint work with me related to the sum-of-squared-logarithms inequality [2], which led me to think about the results mentioned in this brief educational note.

1 Introduction

Let $x, y \in \mathbb{R}^n_+$. We denote by $x \prec_E y$ the partial order

$$e_k(x) \le e_k(y), \quad k = 1, \dots, n-1 \quad \text{and} \quad e_n(x) = e_n(y). \tag{1.1}$$

We call a function $F : \mathbb{R}^n_+ \to \mathbb{R}$ *E-monotone* if

$$x \prec_E y \quad \Longrightarrow \quad F(x) \le F(y). \tag{1.2}$$

The SSLI of Bîrsan et al. [1] is the claim that $F(x) = \sum_i \log^2 x_i$ is E-monotone.

Proposition 1. *If $\psi(x) = \int_0^\infty \log(1 + tx) d\mu(t)$ or $\psi(x) = \int_0^\infty \log(t + x) d\mu(t)$, where $d\mu$ is a nonnegative measure, then $\sum_i \psi(x_i)$ is E-monotone. That is, if $x \prec_E y$, then $\sum_i \psi(x_i) \le \sum_i \psi(y_i)$.*

Proof. We only require $e_k(x) \le e_k(y)$ for $1 \le k \le n$ (indicated by writing $x \preceq_E y$). Thus,

$$\textstyle\sum_{k=0}^n t^k e_k(x) = \prod_{i=1}^n (1 + tx_i) \le \prod_{i=1}^n (1 + ty_i) \qquad \forall t \ge 0 \tag{1.3}$$

$$\textstyle\sum_{k=0}^n t^k e_{n-k}(x) = \prod_{i=1}^n (t + x_i) \le \prod_{i=1}^n (t + y_i) \qquad \forall t \ge 0. \tag{1.4}$$

Taking logarithms, multiplying by $d\mu(t)$, and integrating it then follows that

$$\sum_{i=1}^n \int_0^\infty \log(1 + tx_i) d\mu(t) \le \sum_{i=1}^n \int_0^\infty \log(1 + ty_i) d\mu(t) \implies \textstyle\sum_i \psi(x_i) \le \sum_i \psi(y_i).$$

Similarly, with (1.4) we also obtain $\sum_i \psi(x_i) \le \sum_i \psi(y_i)$. □

[1] Added 18/9/2015: It has been brought to our attention [4] that a line of approach closely related to ours has been developed by M. Šilhavý, who considers a rich generalization based on Pick functions. This idea is natural and elegant (and was also suggested by the author to P. Neff on June 1, 2015. Completely independent of us, M. Šilhavý has recently developed the Pick function approach fully [4]), and parts of it are implicit in the recent article of Jozsa and Mitchison (2015). The key contribution of Šilhavý's work is presentation of necessary and sufficient conditions for the E-monotonicity studied in this paper.

As a corollary we obtain the announced short proof of the SSLI.

Corollary 2. *Let $x, y \in \mathbb{R}^n_+$ such that $x \prec_E y$. Then, $\sum_i \log^2 x_i \leq \sum_i \log^2 y_i$.*

Proof. Our key inspiration is the following integral

$$\log^2 x = \int_0^\infty \log\left(\frac{(1+tx)\,(t+x)}{x(1+t)^2}\right)\frac{dt}{t}. \tag{1.5}$$

As $e_n(x) = e_n(y)$, $\sum_i \log(rx_i) = \sum_i \log(ry_i)$ for any $r > 0$. Combine this with (1.3) and (1.4) to obtain

$$\textstyle\sum_i \log(1+tx_i) + \log(t+x_i) - \log\big((1+t)^2 x_i\big) \leq \sum_i \log(1+ty_i) + \log(t+y_i) - \log\big((1+t)^2 y_i\big).$$

Using (1.5) and integrating both sides over t with $d\mu(t) = \frac{dt}{t}$ the desired claim follows. □

Additional corollaries of Prop. 1 are mentioned below.

Corollary 3. *Let $x, y \in \mathbb{R}^n_+$ such that $x \prec_E y$. Then,*

$$\textstyle\sum_{i=1}^n x_i^p \leq \sum_{i=1}^n y_i^p, \qquad 0 < p < 1, \tag{1.6}$$

$$\textstyle\sum_{i=1}^n x_i^p \geq \sum_{i=1}^n y_i^p, \qquad 1 < p < 2, \quad \textit{and } e_1(x) = e_1(y), \tag{1.7}$$

$$\textstyle-\sum_{i=1}^n x_i \log x_i \leq -\sum_{i=1}^n y_i \log y_i, \quad \textit{if} \quad e_1(x) = e_1(y). \tag{1.8}$$

Proof. To establish (1.6), recall that for $0 < p < 1$ we have the integral representation

$$x^p = \frac{p\sin(p\pi)}{\pi}\int_0^\infty \log(1+tx)t^{-p-1}dt. \tag{1.9}$$

Consequently, an application of Prop 1 immediately yields (1.6).
To establish (1.7), consider the integral representation

$$x^p = \frac{p\sin(p\pi)}{\pi}\int_0^\infty \left(\log(1+tx) - tx\right)t^{-p-1}dt, \tag{1.10}$$

where the integral converges provided $1 < p < 2$ and $x \geq 0$. Since $x \prec_E y$ and we also have $e_1(x) = e_1(y)$, it follows that

$$\textstyle\sum_i \big(\log(1+tx_i) - tx_i\big) \leq \sum_i \big(\log(1+ty_i) - ty_i\big).$$

Thus, using (1.10) and noting that $\sin(p\pi) < 0$ for $1 < p < 2$, we obtain the inequality

$$\textstyle\sum_i x_i^p \geq \sum_i y_i^p.$$

To obtain (1.8) we apply a limiting argument to (1.7). In particular, recall that

$$\lim_{p\to 1}\frac{x_i^p - x_i}{p-1} = x_i \log x_i,$$

so that upon using $\sum_i x_i = \sum_i y_i$ in (1.7), dividing by $p-1$, and taking limits we obtain (1.8). □

We close by noting that apart from (1.3) and (1.4), a more general technique for obtaining an E-monotone function ϕ is to consider

$$\psi(x_1, \ldots, x_n) = \int_0^\infty h\left(\textstyle\prod_{i=1}^n (t+x_i)\right)d\mu(t), \tag{1.11}$$

where h is any monotonic function and μ is a nonnegative measure. As an example, consider the following inequalities that may appear difficult at first sight, but follow easily upon using (1.11) with $h(z) = 1/z$ and $d\mu(t) = t^p dt$,

$$\sum_{i=1}^{n} \frac{(-1)^{i+1} x_i^p}{\prod_{j \neq i}(x_i - x_j)} \quad \geq \quad \sum_{i=1}^{n} \frac{(-1)^{i+1} y_i^p}{\prod_{j \neq i}(y_i - y_j)}, \qquad \text{if } x \prec_E y,\ 0 < p < 1.$$

Finally, note the "subentropy" for a probability vector x [3]

$$Q(x_1, \ldots, x_n) := -\sum_{i=1}^{n} \frac{x_i^n}{\prod_{j \neq i}(x_i - x_j)} \log x_i, \tag{1.12}$$

for which the following "half-axis" formula for Q is derived in [3]:

$$Q = -\int_0^\infty \Big[\frac{t^n}{\prod_{j=1}^n (t + x_j)} - \frac{t}{1+t}\Big] dt. \tag{1.13}$$

From this formula, it is clear that if $x \preceq_E y$ such that $e_1(x) = e_1(y) = 1$, then $Q(x) \leq Q(y)$. We leave an exploration of additional inequalities of this character to the reader.

1.1 Related work

1. Proposed in 2012 [6]; Case $n = 2, 3$ solved by [1]; Then $n = 3$ and its implications in [7]; Much further work solved $n = 4$ [8], but techniques did not extend to general case.
2. First full proof, via complex analysis by L. Borisov on MatheOverflow [5]. A formal writeup outlining details of the proof was presented shortly thereafter in [2]
3. Around the same time (approximately two weeks after L. Borisov's proof) M. Šilhavý characterized E-monotone functions [4]. His results not only imply SSLI, but also provide in fact necessary and sufficient conditions based on the theory of Pick functions; this is a natural and elegant approach, that was foreshadowed in the work of [3], though not pursued the same way.
4. This didactic note (results obtained shortly after) presents an elementary derivation, which yields proof of SSLI as a corollary (Corollary 2). Previous proofs are slightly more intricate.

References

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