A tiny remark on Cauchy-Schwarz via matrices

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Recently, Harvey [2014] observed a version of CS involving four vectors. He noted that although simple, this version is absent in the literature and that its proof requires a simple but additional argument. This tiny note shows that Harvey's inequality actually follows from CS by using an idea based on matrices.

The standard CS inequality is

$$(a^\top b)^2 \le a^\top a b^\top b \qquad \forall\, a, b \in \mathbb{R}^n. \tag{0.1}$$

Harvey [2014] mentions the following four generalizations to (0.1):

(i) $2a^\top bc^\top d \le a^\top ab^\top b + c^\top cd^\top d$ for all $a, b, c, d \in \mathbb{R}^n$

(ii) $2a^\top cb^\top d \le a^\top ab^\top b + c^\top cd^\top d$ for all $a, b, c, d \in \mathbb{R}^n$

(iii) $2a^\top cb^\top d + (a^\top b)^2 + (c^\top d)^2 \le a^\top ab^\top b + c^\top cd^\top d + (a^\top c)^2 + (b^\top d)^2$ for all $a, b, c, d \in \mathbb{R}^n$

(iv) $2a^\top cb^\top d + (a^\top b)^2 + (c^\top d)^2 \le a^\top ab^\top b + c^\top cd^\top d + (a^\top d)^2 + (b^\top c)^2$ for all $a, b, c, d \in \mathbb{R}^n$.

Harvey [2014] notes that although (i), (iii) are immediate from CS, and (ii) is known ([Dragomir, 2003, Thm. 6]), somehow (iv) requires an additional argument. This note shows that actually (iv) also follows from ordinary CS by formulating the inequality using matrices.

The key idea is to recall the following matrix-based proof of CS.

CS VIA MATRICES. Let $A = xy^\top - yx^\top$. Since $\operatorname{tr}(A^\top A) = \sum_{ij} a_{ij}^2 \ge 0$, it immediately follows that $0 \le \operatorname{tr}(A^\top A) = 2x^\top xy^\top y - 2(x^\top y)^2$, which is nothing but CS. □

This matrix-based proof suggests a two-line proof of (iv); here is how.

INEQUALITY (IV) VIA MATRICES. Let $A = ab^\top - ba^\top$ and $B = dc^\top - cd^\top$. Viewing A and B as "stacked" vectors, CS may be expressed as

$$\operatorname{tr}(AB) \le [\operatorname{tr}(A^\top A)]^{1/2}[\operatorname{tr}(B^\top B)]^{1/2}.$$

Expanding the traces, applying AMGM to the right-hand side, and simplifying we obtain

$$2a^\top cb^\top d - 2a^\top db^\top c \le a^\top ab^\top b - (a^\top b)^2 + c^\top cd^\top d - (c^\top d)^2,$$

which immediately implies (iv) since $2a^\top db^\top c \le (a^\top d)^2 + (b^\top c)^2$. □

References

S. S. Dragomir. A survey on Cauchy-Buniakowsky-Schwarz type discrete inequalities. *J. Inequalities in Pure and Applied Mathematics*, 4(3), 2003.

N. J. A. Harvey. A generalization of the Cauchy-Schwarz inequality involving four vectors. *J. Mathematical Inequalities*, 2014. *To appear.*