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## Hlawka–Popoviciu inequalities on positive definite tensors



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### ABSTRACT

We prove inequalities on symmetric tensor sums of positive definite operators. In particular, we prove multivariable operator inequalities inspired by generalizations to the well-known Hlawka and Popoviciu inequalities. As corollaries, we obtain generalized Hlawka and Popoviciu inequalities for determinants, permanents, and generalized matrix functions. The new operator inequalities and their corollaries contain a few recently published inequalities on positive definite matrices as special cases.

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## 1. Introduction

Let  $X$  be a complex inner product space with norm  $\|\cdot\|$ , and let  $a, b, c \in X$  be arbitrary vectors. The inequality

$$\|a + b + c\| + \|a\| + \|b\| + \|c\| \geq \|a + b\| + \|a + c\| + \|b + c\| \tag{1.1}$$

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is known as *Hlawka’s inequality*. It seems to have appeared first in a paper of Hor-nich [7] (who credits the proof to Hlawka). Several proofs are known, see e.g. [14, p. 100] or [13, pp. 171–172]. This inequality has witnessed a long series of investigations and generalizations—we refer the reader to the recent work of Fechner [5] for an excellent summary of related work as well as a substantial list of references. Fechner himself con-siders the *functional Hlawka inequality*

$$f(a + b + c) + f(a) + f(b) + f(c) \geq f(a + b) + f(a + c) + f(b + c), \tag{1.2}$$

and studies real valued functions  $f$  on an abelian group  $(A, +)$  that satisfy (1.2).

To our knowledge, all authors who previously published Hlawka type inequalities limited their attention to inequalities over the reals. In contrast, we study “operator Hlawka inequalities,” so that instead of the total order on the reals, we consider the Löwner partial order ‘ $\geq$ ’ on Hermitian positive definite matrices or operators. As a consequence, we are able to recover as corollaries several Hlawka type inequalities for scalar valued matrix functions known as “generalized matrix functions,” which include the determinant and permanent as special cases. Observe also the resemblance between (1.2) and *Popoviciu’s inequality* (1.3), which states for a convex function  $f$  on a real interval  $I$  and  $a, b, c \in I$  that

$$3f\left(\frac{a+b+c}{3}\right) + f(a) + f(b) + f(c) \geq 2\left(f\left(\frac{a+b}{2}\right) + f\left(\frac{a+c}{2}\right) + f\left(\frac{b+c}{2}\right)\right). \tag{1.3}$$

In fact, this resemblance will allow us to obtain some operator Popoviciu inequalities. After having submitted this paper, it was brought to the authors’ attention that some results of the paper are also obtained independently in [2].

*Notation and background*

Throughout this paper, matrices and tensors are denoted by upper case letters. Unless otherwise specified, all matrices are assumed to be of same size (say  $m \times m$ ), self-adjoint and positive (semi)definite. The operator inequality  $A \geq B$  denotes the Löwner partial order, meaning that  $A - B \geq 0$  is positive definite. Wherever multiplication is used, we mean tensor products (though unusual, we use this notation for aesthetic reasons to keep the “visual burden” of our proofs low); thus for arbitrary matrices  $A, B$ :

$$A^p \equiv A^{\otimes p} = A \otimes A \otimes \cdots \otimes A \quad (p \text{ times})$$

$$A^p B^q \equiv (A^{\otimes p}) \otimes (B^{\otimes q}) \quad (\text{integers } p, q).$$

Note that this multiplication is noncommutative, so  $AB \neq BA$ .

We write  $[n]$  to denote the set  $\{1, 2, \dots, n\}$  and  $[n \setminus k]$  to denote the set  $[n] \setminus \{k\}$ .

For some indices, we use MATLAB notation, e.g. the form  $i = 1 : 2 : 2k - 1$  meaning that  $i$  increases in steps of 2, taking on only the values  $1, 3, 5, \dots, 2k - 1$ .

Since the entire paper relies extensively on elementary properties of Kronecker (tensor) products, let us briefly recall these below.

**Proposition 1.1.** *Let  $A, B, C, D$  be positive definite operators. Then,*

- (i)  $AB \equiv A \otimes B$  is also positive definite
- (ii) If  $A \geq B$  and  $C \geq D$  then  $AC \geq BD$
- (iii)  $A(B + C) = AB + AC$ ,  $(A + B)C = AC + BC$
- (iv)  $(A + B)^p \geq A^p + B^p$  for all  $p \in \mathbb{N}$ .

## 2. Hlawka type inequalities for three operators

With this background we are ready to prove our first operator Hlawka inequality.

**Theorem 2.1.** *Let  $A, B, C$  be positive definite operators. Then for each integer  $p \geq 1$ ,*

$$(A + B + C)^p + A^p + B^p + C^p \geq (A + B)^p + (A + C)^p + (B + C)^p. \tag{2.1}$$

**Proof.** The case  $p = 1$  is trivial and holds with equality. Unsurprisingly, for  $p = 2$  we again have equality, since both sides expand to

$$2(A^2 + B^2 + C^2) + AB + BA + AC + CA + BC + CB.$$

We prove the general claim by induction. Assume therefore that (2.1) holds for some integer  $p \geq 2$ . Then,

$$\begin{aligned} (A + B + C)^{p+1} &= (A + B + C)^p(A + B + C) \\ &\geq ((A + B)^p + (A + C)^p + (B + C)^p - A^p - B^p - C^p)(A + B + C) \\ &= (A + B)^{p+1} + (A + C)^{p+1} + (B + C)^{p+1} \\ &\quad - A^{p+1} - B^{p+1} - C^{p+1} + \mathcal{T}, \end{aligned}$$

where the inequality follows from the induction hypothesis. The term  $\mathcal{T}$  is defined as

$$\mathcal{T} = (A + B)^p C + (A + C)^p B + (B + C)^p A - A^p(B + C) - B^p(A + C) - C^p(A + B).$$

It remains to show that  $\mathcal{T} \geq 0$ . But this follows immediately upon applying the superadditivity inequality Proposition 1.1(iv) to the first three terms of  $\mathcal{T}$  and canceling. Thus, inequality (2.1) is proved.  $\square$

Theorem 2.1 yields the following result of Tie et al. [17, Lemma 2.2] as a corollary. (Note that the inequality (2.2) is called *strong superadditivity* of tensor products; readers familiar with combinatorics may recognize it as *supermodularity*.)

**Corollary 2.2.** *Let  $A, B, C$  be positive definite operators. Then for each integer  $p \geq 1$ ,*

$$(A + B + C)^p + A^p \geq (A + B)^p + (A + C)^p. \tag{2.2}$$

**Proof.** Immediate upon combining Proposition 1.1(iv) with inequality (2.1).  $\square$

Using the operator inequality (2.1) and restricting to suitable symmetry classes, we can obtain Hlawka inequalities for determinants, permanents, and immanants. This line of thought is well known in matrix analysis, see e.g. [1, p. 114] and also [15].

Specifically, let  $G$  be a subgroup of the symmetric group  $\mathfrak{S}_m$  on  $m$  letters, and let  $\chi$  be an irreducible character of  $G$ . The  $G$ -*immanant* (also known as *generalized matrix function* [11,12]) of an arbitrary  $m \times m$  complex matrix  $X$  is defined as

$$d_\chi^G(X) := \sum_{\sigma \in G} \chi(\sigma) \prod_{i=1}^m a_{i,\sigma(i)}. \tag{2.3}$$

When  $G = \mathfrak{S}_m$  and  $\chi(\sigma) = \text{sgn}(\sigma)$  we have  $d_\chi^G(X) = \det(X)$ ;  $\chi(\sigma) \equiv 1$  yields the permanent, while other choices yield immanants [11]. Using arguments from multilinear algebra (e.g., [11,8]), it can be shown [11, p. 126] that there exists a matrix  $Z_{G,\chi}$  such that

$$d_\chi^G(X) = Z_{G,\chi}^*(\otimes^m X)Z_{G,\chi}. \tag{2.4}$$

Using representation (2.4) and Theorem 2.1 we then obtain the following corollary.

**Corollary 2.3.** *Let  $A, B, C$  be positive definite, and let  $d_\chi^G$  be as defined by (2.3). Then,*

$$d_\chi^G(A + B + C) + d_\chi^G(A) + d_\chi^G(B) + d_\chi^G(C) \geq d_\chi^G(A + B) + d_\chi^G(A + C) + d_\chi^G(B + C). \tag{2.5}$$

**Proof.** Congruence preserves Löwner partial order, so we use (2.4) and (2.1) and conclude.  $\square$

**Remark 2.4.** The recent strong superadditivity result of Paksoy et al. [15, Theorem 3.2] for three matrices follows by combining (2.4) with Corollary 2.3 and Proposition 1.1(iv).

**Remark 2.5.** M. Lin brought to our notice his very recent result that establishes inequality (2.5) for the special case of determinants [9]. His proof uses only elementary methods, is entirely different from our approach, and is of instructive value.

### 3. A multivariable tensor Hlawka inequality

It turns out that the above results can be obtained as corollaries of a more general operator inequality involving  $n$  positive definite matrices. Before considering this more

general inequality, let us mention a Hlawka type inequality that was conjectured by the first named author, which originally inspired this paper.

**Conjecture 3.1** (Berndt). For  $n \geq 3$ , let  $A_1, \dots, A_n$  be positive definite; for each  $k = 1, \dots, n$ , let  $s_k$  be the elementary symmetric determinantal polynomial

$$s_k := \sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \det(A_{i_1} + \dots + A_{i_k}).$$

Then, the following generalization of the Hlawka inequality holds:

$$s_n + s_{n-2} + \dots \geq s_{n-1} + s_{n-3} + \dots. \tag{3.1}$$

Inequality (3.1) may come as a surprise to those who study Hlawka type inequalities. Indeed, Freudenthal [6] considered generalizing the basic norm inequality (1.1) to a form similar to (3.1). Specifically, he asked whether for  $n$  vectors  $a_1, \dots, a_n$  the inequality

$$\sum_{i=1}^n \|a_i\| - \sum_{i < j} \|a_i + a_j\| \pm \dots + (-1)^{n-1} \|a_1 + \dots + a_n\| \geq 0$$

holds. According to Mitrinović [13, p. 174], this inequality was shown to be false for  $n \geq 4$  by W.A.J. Luxemburg. Nevertheless, other multivariable generalizations do hold, among which the following seems to be of the most general kind:

**Proposition 3.2.** (See [16, Corollary 3.5].) Let  $H$  be a metric space,  $n \geq 3$  and  $k \in \{2, \dots, n\}$ . Then for all  $a_1, \dots, a_n \in H$ ,

$$\sum_{1 \leq i_1 < i_2 < \dots < i_k \leq n} \|a_{i_1} + \dots + a_{i_k}\| \leq \binom{n-2}{k-1} \sum_{i=1}^n \|a_i\| + \binom{n-2}{k-2} \left\| \sum_{i=1}^n a_i \right\|.$$

We now proceed to show that for positive operators a multivariable Hlawka type inequality does hold. Combined with representation (2.4), it then implies not only the determinantal inequality (3.1) but also its  $G$ -immanant version.

For positive integers  $k, n, p$  with  $k \leq n$  define the following symmetric tensor sums:

$$S_{k,[n]}^p := \sum_{I \subseteq [n], |I|=k} \left( \sum_{i \in I} A_i \right)^p. \tag{3.2}$$

The main result of this paper is the following theorem.

**Theorem 3.3.** Let  $n \geq 3$  and  $A_1, \dots, A_n \geq 0$ . Then, for  $p \in \mathbb{N}$  the operator inequality

$$S_{n,[n]}^p + S_{n-2,[n]}^p + \dots \geq S_{n-1,[n]}^p + S_{n-3,[n]}^p + \dots \tag{3.3}$$

holds.

**Proof.** We prove the claim (call it  $C_{n,p}$ ) by induction on  $n$ . For  $n = 3$ ,  $C_{3,p}$  is the Hlawka inequality established by [Theorem 2.1](#). Fix  $n \geq 4$  and suppose we have proved  $C_{n-1,p}$  for all  $p$ . We first assume that  $n$  is even (the argument for odd  $n$  will be similar).

We now perform an induction on  $p$ . For  $p = 1$ , the claim clearly holds as both sides of [\(3.3\)](#) are equal. Assume therefore that the claim holds up to some integer  $p - 1$ . Thus,

$$S_{n,[n]}^{p-1} + S_{n-2,[n]}^{p-1} + \dots + S_{2,[n]}^{p-1} \geq S_{n-1,[n]}^{p-1} + S_{n-3,[n]}^{p-1} + \dots + S_{1,[n]}^{p-1}.$$

Multiplying (i.e., taking tensor products) both sides by  $(A_1 + \dots + A_n)$  on the right and using [Proposition 1.1\(ii\)](#), we obtain

$$\sum_{j=2:2:n} S_{j,[n]}^p + \mathcal{L} \geq \sum_{j=1:2:n-1} S_{j,[n]}^p + \mathcal{R},$$

where  $\mathcal{L}$  and  $\mathcal{R}$  denote the respective mixed terms. The claim  $C_{n,p}$  will be proved if we show that  $\mathcal{R} \geq \mathcal{L}$ . Details follow below.

An easy rearrangement of the respective terms shows that

$$\begin{aligned} \mathcal{L} &= \sum_{\substack{I \subset [n] \\ |I|=n-2}} \left( \sum_{i \in I} A_i \right)^{p-1} \left( \sum_{i \notin I} A_i \right) + \dots + \sum_{\substack{I \subset [n] \\ |I|=2}} \left( \sum_{i \in I} A_i \right)^{p-1} \left( \sum_{i \notin I} A_i \right) \\ \mathcal{R} &= \sum_{\substack{I \subset [n] \\ |I|=n-1}} \left( \sum_{i \in I} A_i \right)^{p-1} \left( \sum_{i \notin I} A_i \right) + \dots + \sum_{k=1}^n A_k^{p-1} \left( \sum_{i \neq k} A_i \right) \end{aligned} \tag{3.4}$$

Note that the main sums in  $\mathcal{L}$  and  $\mathcal{R}$  are only over even and odd sized subsets, respectively.

The key to the proof is the following regrouping of [\(3.4\)](#), which reveals the underlying inductive structure:

$$\begin{aligned} \mathcal{R} &= \left( \sum_{i \in [n \setminus n]} A_i \right)^{p-1} A_n + \left( \sum_{\substack{I \subset [n \setminus n] \\ |I|=n-3}} \left( \sum_{i \in I} A_i \right)^{p-1} \right) A_n + \dots + \left( \sum_{\substack{I \subset [n \setminus n] \\ I=\{i\}}} A_i^{p-1} \right) A_n \\ &+ \left( \sum_{i \in [n \setminus n-1]} A_i \right)^{p-1} A_{n-1} + \left( \sum_{\substack{I \subset [n \setminus n-1] \\ |I|=n-3}} \left( \sum_{i \in I} A_i \right)^{p-1} \right) A_{n-1} + \dots \\ &+ \left( \sum_{\substack{I \subset [n \setminus n-1] \\ I=\{i\}}} A_i^{p-1} \right) A_{n-1} \\ &+ \dots + \dots + \dots + \dots \\ &+ \left( \sum_{i \in [n \setminus 1]} A_i \right)^{p-1} A_1 + \left( \sum_{\substack{I \subset [n \setminus 1] \\ |I|=n-3}} \left( \sum_{i \in I} A_i \right)^{p-1} \right) A_1 + \dots + \left( \sum_{\substack{I \subset [n \setminus 1] \\ I=\{i\}}} A_i^{p-1} \right) A_1, \end{aligned}$$

and

$$\begin{aligned} \mathcal{L} = & \left( \sum_{\substack{I \subset [n \setminus n] \\ |I|=n-2}} A_i \right)^{p-1} A_n + \left( \sum_{\substack{I \subset [n \setminus n] \\ |I|=n-4}} \left( \sum_{i \in I} A_i \right)^{p-1} \right) A_n + \cdots + \left( \sum_{\substack{I \subset [n \setminus n] \\ |I|=2}} A_i^{p-1} \right) A_n \\ & + \cdots + \cdots + \cdots + \cdots \\ & + \left( \sum_{\substack{I \subset [n \setminus 1] \\ |I|=n-2}} A_i \right)^{p-1} A_1 + \left( \sum_{\substack{I \subset [n \setminus 1] \\ |I|=n-4}} \left( \sum_{i \in I} A_i \right)^{p-1} \right) A_1 + \cdots + \left( \sum_{\substack{I \subset [n \setminus 1] \\ |I|=2}} A_i^{p-1} \right) A_1. \end{aligned}$$

The above expressions may be more succinctly written as

$$\begin{aligned} \mathcal{R} = & \left( \sum_{j=1:2:n-1} S_{j,[n \setminus n]}^{p-1} \right) A_n + \left( \sum_{j=1:2:n-1} S_{j,[n \setminus n-1]}^{p-1} \right) A_{n-1} + \cdots + \left( \sum_{j=1:2:n-1} S_{j,[n \setminus 1]}^{p-1} \right) A_1 \\ \mathcal{L} = & \left( \sum_{j=2:2:n-2} S_{j,[n \setminus n]}^{p-1} \right) A_n + \left( \sum_{j=2:2:n-2} S_{j,[n \setminus n-1]}^{p-1} \right) A_{n-1} + \cdots + \left( \sum_{j=2:2:n-2} S_{j,[n \setminus 1]}^{p-1} \right) A_1. \end{aligned}$$

For each pair of corresponding terms between  $\mathcal{R}$  and  $\mathcal{L}$ , we can apply the statement  $C_{n-1,p-1}$  because each set  $[n \setminus k]$  is of size  $n - 1$ . So we conclude that  $\mathcal{R} \geq \mathcal{L}$ .

If  $n$  is odd, the only difference is in the indices of the summations, which now run over  $j = 1 : 2 : n - 2$  for  $\mathcal{L}$  and  $j = 2 : 2 : n - 1$  for  $\mathcal{R}$ . We conclude again that  $\mathcal{R} \geq \mathcal{L}$ , finishing the proof.  $\square$

**Corollary 3.4.** *Conjecture 3.1 is true.*

**Proof.** Recall that for an  $m \times m$  matrix  $A$ ,  $\det(A) = \wedge^m A$ , where  $\wedge$  denotes the usual (Grassmann) exterior product. Moreover, there exists a matrix  $Z$  such that  $\wedge^m A = Z^*(A^{\otimes m})Z$ . Since congruence preserves Löwner order, setting  $p = m$  in (3.3) and transforming with  $Z$ , we immediately obtain inequality (3.1).  $\square$

Using the argument of Corollary 3.4 along with (2.4), we obtain a more general result.

**Corollary 3.5.** *Conjecture 3.1 is true even when determinants are replaced by  $G$ -immanants.*

We note in passing that even more is true: Combining Theorem 3.3 with the proof technique of [10] we can obtain a block-matrix version of inequality (3.3). Specifically, for  $1 \leq i \leq m$  let  $\mathbf{A}_i = [(A_i)_{pq}]_{p,q=1}^m \geq 0$  be positive definite block matrices comprised of  $d \times d$  complex matrices  $(A_i)_{pq}$ . Define  $d_\chi^G(\mathbf{A}) := [d_\chi^G(A_{pq})]_{p,q=1}^m$  for a block matrix  $\mathbf{A}$ . Then, Corollary 3.5 holds in its “completely positive” incarnation applied to a collection of block matrices  $\mathbf{A}_1, \dots, \mathbf{A}_n$ . We leave the details as an exercise for the interested reader.

#### 4. From Popoviciu to Hlawka

In this section we explore the connection with Popoviciu type inequalities alluded to in the introduction. In particular, we follow the proof technique of [Theorem 3.3](#) to establish several Popoviciu type inequalities, one of which recovers the multivariable  $G$ -immanant “superadditivity” inequality of [[15](#), [Theorem 4.1](#)] as a special case.

To simplify notation, we will frequently drop subscripts on summations; hence  $\sum$  is understood to mean  $\sum_{i=1}^n$  or  $\sum_{k=1}^n$ , the choice being clear from context.

For a convex function  $f : \mathbb{R} \rightarrow \mathbb{R}$  and scalars  $x_1, \dots, x_k$  Jensen’s inequality says that

$$f(x_1) + \cdots + f(x_k) \geq kf\left(\frac{x_1 + \cdots + x_k}{k}\right).$$

After Jensen’s inequality, Popoviciu’s inequality may be considered as the next-to-simplest inequality for convex functions. We restate it here.

**Proposition 4.1.** *If  $f$  is a convex function on a real interval  $I$  and  $x_1, x_2, x_3 \in I$ , then*

$$f(x_1) + f(x_2) + f(x_3) + 3f\left(\frac{x_1+x_2+x_3}{3}\right) \geq 2\left(f\left(\frac{x_1+x_2}{2}\right) + f\left(\frac{x_1+x_3}{2}\right) + f\left(\frac{x_2+x_3}{2}\right)\right) \quad (4.1)$$

Formally, inequality [\(4.1\)](#) resembles Hlawka’s inequality (up to scaling factors, which are actually crucial). This resemblance motivates us to examine if some known generalizations to Popoviciu’s inequality for scalars also extend to positive operators.

We begin with the following generalization of [\(4.1\)](#) given by Cirtoaje [[3](#)].

**Proposition 4.2.** *Let  $f$  be convex on a real interval  $I$ , and  $x_1, x_2, \dots, x_n \in I$ . Then,*

$$f(x_1) + \cdots + f(x_n) + \frac{n}{n-2}f\left(\frac{x_1 + \cdots + x_n}{n}\right) \geq \frac{2}{n-2} \sum_{i < j} f\left(\frac{x_i + x_j}{2}\right).$$

Comparing the case  $n = 3$  [\(4.1\)](#) with the Hlawka inequality [\(1.1\)](#), it is clear that  $kf\left(\frac{x_1 + \cdots + x_k}{k}\right)$  should correspond to  $\|a_1 + \cdots + a_k\|$ . In terms of tensor sums, after multiplying with  $(n - 2)$ , we are led to conjecture [\(4.2\)](#), which turns out to be true.

**Theorem 4.3.** *Let  $A_1, \dots, A_n$  be positive definite operators. Then for each integer  $p \geq 1$ ,*

$$(n - 2) \sum A_i^p + \left(\sum A_i\right)^p \geq \sum_{i < j} (A_i + A_j)^p. \quad (4.2)$$

**Proof.** The proof is similar to the one of [Theorem 2.1](#). We proceed by induction on  $p$ . For  $p = 1$ , both sides of [\(4.2\)](#) are equal to  $(n - 1) \sum A_i$ ; for  $p = 2$  we again have equality,



since both sides of (4.2) are equal to

$$(n - 1) \sum A_i^2 + \sum_{i < j} (A_i A_j + A_j A_i).$$

Assume for the inductive step that (4.2) holds for some integer  $p \geq 2$ . Then for  $p + 1$ ,

$$\begin{aligned} & (\sum A_i)^{p+1} + (n - 2) \sum A_i^{p+1} \\ &= \underbrace{\left( \sum A_i \right)^p \left( \sum A_k \right)} + (n - 2) \sum A_i^p \sum A_k - (n - 2) \sum_{i < j} (A_i^p A_j + A_j^p A_i) \\ &\geq \sum_{i < j} (A_i + A_j)^p \sum A_k - (n - 2) \sum_{i < j} (A_i^p A_j + A_j^p A_i) \\ &= \sum_{i < j} (A_i + A_j)^{p+1} + \sum_{i < j} (A_i + A_j)^p \sum_{k \notin \{i, j\}} A_k - (n - 2) \sum_{i < j} (A_i^p A_j + A_j^p A_i) \\ &\geq \sum_{i < j} (A_i + A_j)^{p+1} + \sum_{i < j} (A_i^p + A_j^p) \sum_{k \notin \{i, j\}} A_k - (n - 2) \sum_{i < j} (A_i^p A_j + A_j^p A_i) \\ &= \sum_{i < j} (A_i + A_j)^{p+1}. \end{aligned}$$

The first inequality follows from the induction hypothesis applied to the underbraced term, while the second inequality follows from superadditivity Proposition 1.1(iv). The final equality is easy to verify: Fix  $i = 1$ , then the second term yields for each  $j = 2, \dots, n$  the product of  $A_1^p$  with  $(n - 2)$  of the  $A_k$ 's ( $k \neq 1$ ), so for each  $k \neq 1$  the product  $A_1^p A_k$  occurs  $(n - 2)$  times, and so it does also in the negative term. By symmetry, the same holds for all  $i$ . □

**Corollary 4.4.** *Let  $A_1, \dots, A_n$  be positive definite, and let  $d_\chi^G$  be as in (2.3). Then,*

$$(n - 2) \sum d_\chi^G(A_i) + d_\chi^G(\sum A_i) \geq \sum_{i < j} d_\chi^G(A_i + A_j).$$

Corollary 4.4 combined with the superadditivity inequality Proposition 1.1(iv) for the appropriate pairs of indices implies the following result of Paksoy et al. [15].

**Corollary 4.5.** *(See [15, Theorem 4.1].) Let  $A_1, \dots, A_n$  and  $d_\chi^G$  be as in Corollary 4.4. Then,*

$$d_\chi^G(A_1 + \dots + A_n) \geq \sum_{i \neq j} d_\chi^G(A_i + A_j) - (n - 2)d_\chi^G(A_i) \quad \text{for each } i = 1, \dots, n.$$

Before stating the most general result in this direction, we mention an intermediate generalization of Popoviciu’s inequality, which we call *Popoviciu–Cirtoaje–Zhao inequality*.<sup>1</sup> It states the following:

**Proposition 4.6.** *If  $f$  is a convex function on a real interval  $I$  and  $x_1, x_2, \dots, x_n \in I$ , then for  $2 \leq m < n$ ,*

$$\binom{n-2}{m-1} (f(x_1) + \dots + f(x_n)) + n \binom{n-2}{m-2} f\left(\frac{x_1 + \dots + x_n}{n}\right) \geq m \sum_{i_1 < \dots < i_m} f\left(\frac{x_{i_1} + \dots + x_{i_m}}{m}\right)$$

The corresponding generalization of [Theorem 4.3](#) is

**Theorem 4.7.** *Let  $A_1, \dots, A_n$  be positive definite operators. Then for each integer  $p \geq 1$ ,*

$$\binom{n-2}{m-1} \sum A_i^p + \binom{n-2}{m-2} \left(\sum A_i\right)^p \geq \sum_{i_1 < \dots < i_m} (A_{i_1} + \dots + A_{i_m})^p. \tag{4.3}$$

Instead of proving [Theorem 4.7](#), we move on to the most general Popoviciu type inequality for tensors, motivated by a scalar case partially treated in [\[4\]](#).

**Theorem 4.8.** *Let  $A_1, \dots, A_n$  be positive definite operators. Let  $S_{k,[n]}^p$  be defined as in [\(3.2\)](#). Then for integers  $1 \leq k < \ell < m \leq n$ ,*

$$\frac{m-\ell}{k \binom{n}{k}} S_{k,[n]}^p + \frac{\ell-k}{m \binom{n}{m}} S_{m,[n]}^p \geq \frac{m-k}{\ell \binom{n}{\ell}} S_{\ell,[n]}^p.$$

We omit the proof for brevity; it can be obtained by following the inductive technique developed above. It should be mentioned that the corresponding inequality for convex functions only holds for certain choices of  $k, \ell, m$  [\[4\]](#).

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<sup>1</sup> This scalar inequality was proposed by Yufei Zhao (username Billzhao) and Vasile Cirtoaje (username Vasc) on the website AoPS [\[3\]](#) and was soon proved by Darij Grinberg in the same thread.

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