Note #4. The Determinant Kernel

Author: Suvrit Sra
ILAS IMAGE # 51
Creation Date: Apr. 24, 2012
Publication date: Nov 2013.

Let $X_1, X_2, \ldots, X_n$ be $p \times p$ symmetric positive definite matrices. Let K be given by

$$K = [k_{ij}]_{i,j=1}^{n} = \left[\frac{1}{\det(X_i + X_j)}\right]. \tag{4.1}$$

Prove that K is positive definite.

Let $\beta > \frac{p-1}{2}$ be a scalar. A harder problem is to prove that the matrix

$$K_\beta = [k_{ij}^{\beta}] = \left[\frac{1}{\det\left((X_i + X_j)^{\beta}\right)}\right], \tag{4.2}$$

is also positive definite.

To prove that K given by (4.1) is positive definite, recall the Gaussian integral

$$\int_{\mathbb{R}^p} e^{-x^T X x} dx = \pi^{p/2} \det(X)^{-1/2}.$$

Now define the function $f_i := \frac{1}{\pi^{p/4}} e^{-x^T X_i x}$ for $1 \leq i \leq n$. Then, $f_i \in L_2(\mathbb{R}^p)$, and

$$k_{ij} = \langle f_i,\, f_j \rangle := \frac{1}{\pi^{d/2}} \int_{\mathbb{R}^d} e^{-x^T (X_i + X_j) x} dx = \det(X_i + X_j)^{-1/2},$$

which shows that $K_{1/2}$ is positive. From the Schur product theorem we know that the elementwise product of two positive matrices is again positive. So, in particular K_β is positive whenever β is an integer multiple of $1/2$. To extend the result to all $\beta > (n-1)/2$, we invoke another integral representation; viz. the *multivariate Gamma function*, defined as

$$\Gamma_n(\beta) := \int_{A>0} e^{-\operatorname{tr}(A)} \det(A)^{\beta - (p+1)/2} dA, \qquad \text{where } \beta > (p-1)/2, \tag{4.3}$$

where the integration is the set of $p \times p$ positive matrices. Define now the function $f_i := c e^{-\operatorname{tr}(AX_i)}$ for $1 \leq i \leq n$. Then, $f_i \in L_2(\mathbb{P}_p)$, where $\mathbb{P}_p$ is the set of $p \times p$ positive matrices. In this space, we see that the inner product equals

$$k_{ij} = \langle f_i,\, f_j \rangle := c_1 \int_{A>0} e^{-\operatorname{tr}(A(X_i + X_j))} \det(A)^{\beta - (n+1)/2} dA = \det(X_i + X_j)^{-\beta},$$

which exists whenever $\beta > (n-1)/2$. This proves positive definiteness of (4.2).