Note #4. A determinant inequality

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ILAS IMAGE # 49
Creation Date: 3rd May, 2012
Publication date: Dec 2012.

Let $A, B > 0$; let $1 \le t \le u$. Then,

$$\det\left(\frac{A^t + B^t}{2}\right)^{1/t} \le \det\left(\frac{A^u + B^u}{2}\right)^{1/u}. \tag{4.1}$$

Proof. Using $P = A^{-1}$, and $S = B$. We must show that

$$\prod_{j=1}^{n}\left(\frac{1+\lambda_j(P^T S^t)}{2}\right)^{1/t} \le \prod_{j=1}^{n}\left(\frac{1+\lambda_j(P^u S^u)}{2}\right)^{1/u}.$$

We know that (see example, Theorem IX.2.9 of *Matrix Analysis* by R. Bhatia):

$$\lambda^{1/t}(P^t S^t) \prec_{\log} \lambda^{1/u}(P^u S^u),$$

which immediately implies that

$$\prod_{j=1}^{k}\left(\frac{1+\lambda_j^{u/t}(P^t S^t)}{2}\right)^{1/u} \le \prod_{j=1}^{k}\left(\frac{1+\lambda_j(P^u S^u)}{2}\right)^{1/u}, \quad 1 \le k \le n.$$

But, for $u \ge t$, the map $x \mapsto x^{u/t}$ is convex, whereby we have

$$\prod_{j=1}^{k}\left(\frac{1+\lambda_j^{u/t}(P^t S^t)}{2}\right)^{1/u} \ge \prod_{j=1}^{k}\left(\left(\frac{1+\lambda_j(P^t S^t)}{2}\right)^{u/t}\right)^{1/u} = \prod_{j=1}^{k}\left(\frac{1+\lambda_j(P^t S^t)}{2}\right)^{1/t}. \tag{4.2}$$

□