Note #2. ILAS Image 47 Problems

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Question 2.1 (Problem 47-5; Roger Horn)**.** Let $A_1, \ldots, A_N$ be Hermitian positive definite. Let $B = (\sum_i A_i)(\sum_i A_i^{-1})$. Show that B is diagonalizable, has real eigenvalues, and the smallest eigenvalue is not less than N^2.

To ease notation, we use lower case letters. Since b is a product of two positive definite matrices, it has in fact positive real eigenvalues. Also, we may write $b = x^2y^2$, or equivalently, $x^{-1}bx = xy^2x$; but the latter matrix is symmetric, hence diagonalizable. This implies diagonalizability of b.

Now we prove that $\lambda_{\min}(b) \geq N^2$. To that end, recall that the arithmetic mean for positive operators dominates (in Löwner order) the operator Harmonic mean (this fact follows e.g., from operator convexity of $x \mapsto x^{-1}$). This fact implies in particular that

$$\frac{a_1 + \cdots + a_N}{N} \geq N(a_1^{-1} + \cdots + a_N^{-1})^{-1} \implies \big(\sum\nolimits_i a_i\big) \geq N^2\big(\sum\nolimits_i a_i^{-1}\big)^{-1}. \tag{2.1}$$

Now appealing to $\lambda_i(x^{1/2}ax^{1/2}) = \lambda_i(ax)$, after multiplying both sides of (2.2) by $x = (\sum_i a_i^{-1})^{1/2}$ on the left and the right, we obtain the desired result about $\lambda_{\min}(b)$.

Question 2.2 (Problem 47-7; Ramazan Turkmen and Fuzhen Zhang)**.** Let A and B be arbitrary complex matrices. Prove that

$$\lambda_i(A^*A + B^*B + A^*B + B^*A) \leq 2\lambda_i(AA^* + BB^*). \tag{2.2}$$

We use lower case letters for matrices. Define the Hermitian matrices

$$x = \begin{bmatrix} 0 & a \\ a^* & a \end{bmatrix}, \quad y = \begin{bmatrix} 0 & b \\ b^* & 0. \end{bmatrix} \tag{2.3}$$

Recall now that the map $t \to t^2$ is operator convex. Thus, we have

$$\left(\frac{x+y}{2}\right)^2 \leq \frac{x^2+y^2}{2} \quad \leftrightarrow \quad (x+y)^2 \leq 2(x^2+y^2),$$

which in particular implies that

$$\begin{bmatrix} (a+b)(a+b)^* & 0 \\ 0 & (a+b)^*(a+b) \end{bmatrix} \leq \begin{bmatrix} 2(aa^*+bb^*) & 0 \\ 0 & 2(a^*a+b^*b) \end{bmatrix}.$$

This inequality immediately yields

$$\lambda_i((a+b)(a+b)^*) \leq \lambda_i(2(aa^*+bb^*)).$$

But since for any matrix x, the matrices x^*x and xx^* have the same eigenvalues, we conclude that

$$\lambda_i((a+b)^*(a+b)) = \lambda_i((a+b)(a+b)^*) \leq 2\lambda_i((aa^*+bb^*)).$$