Analysis and Design of Optimization Algorithms via Integral Quadratic Constraints
Based on papers by Lessard, Packard, Recht, Nishihara, Jordan

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Context

- Balance robustness, accuracy, speed
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- Current: analyze methods algorithm-by-algorithm
Balance robustness, accuracy, speed
Current: analyze methods algorithm-by-algorithm
Reliance on optimization experts for proofs
Main idea

- Frame first-order methods as dynamical systems
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- Replace nonlinear parts with integral quadratic constraints (IQC)
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- Replace nonlinear parts with integral quadratic constraints (IQC)
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- Optimize over algorithm parameters for convergence rate
  - Subject to strong convexity and Lipschitz properties
  - Subject to extent of noise
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Linear dynamical systems

\[
\begin{align*}
\xi_{k+1} &= A\xi_k + Bu_k \\
y_k &= C\xi_k + Du_k
\end{align*}
\]  

\[(u_k, y_k, \xi_k) = \text{input, output, state}\]
Linear dynamical systems (with nonlinear feedback)

\[ \xi_{k+1} = A\xi_k + Bu_k \quad (3) \]
\[ y_k = C\xi_k + Du_k \quad (4) \]
\[ u_k = \Delta(y_k) \quad (5) \]

\((u_k, y_k, \xi_k) = \text{input, output, state}\)
\(\Delta = (\text{nonlinear}) \text{ map}\)
Linear dynamical systems (for first order methods)

\[ \xi_{k+1} = A\xi_k + Bu_k \]  
\[ y_k = C\xi_k + Du_k \]  
\[ u_k = \Delta(y_k) \]

\[(u_k, y_k, \xi_k) = \text{input, output, state}\]
\[\Delta(z) = \nabla f(z)\]
Gradient descent

- Start with gradient descent update:

\[ x_{k+1} = x_k - \alpha \nabla f(x_k) \]
Gradient descent

- Start with gradient descent update:

\[ x_{k+1} = x_k - \alpha \nabla f(x_k) \]

- Expand to input, output, state:

\[ \xi_{k+1} = \xi_k - \alpha u_k \]
\[ y_k = \xi_k \]
\[ u_k = \nabla f(y_k) \]
Gradient descent

- Start with gradient descent update:
  \[ x_{k+1} = x_k - \alpha \nabla f(x_k) \]

- Expand to input, output, state:
  \[ \xi_{k+1} = \xi_k - \alpha u_k \]
  \[ y_k = \xi_k \]
  \[ u_k = \nabla f(y_k) \]

- Block form:
  \[
  \begin{bmatrix}
  A & B \\
  C & D \\
  \end{bmatrix}
  =
  \begin{bmatrix}
  l_d & -\alpha l_d \\
  l_d & 0_d \\
  \end{bmatrix}
  \]
Nesterov’s method

- Start with update:

\[
x_{k+1} = y_k - \alpha_k \nabla f(y_k)
\]

\[
y_k = (1 + \beta)x_k - \beta x_{k-1}
\]
Nesterov’s method

- Start with update:

$$x_{k+1} = y_k - \alpha_k \nabla f(y_k)$$
$$y_k = (1 + \beta)x_k - \beta x_{k-1}$$

- Expand to input, output, state:

$$\xi^{(1)}_{k+1} = (1 + \beta)\xi^{(1)}_k - \beta \xi^{(2)}_k - \alpha u_k$$
$$\xi^{(2)}_{k+1} = \xi^{(1)}_k$$

$$y_k = (1 + \beta)\xi^{(1)}_k - \beta \xi^{(2)}_k$$
$$u_k = \nabla f(y_k)$$
Nesterov’s method

- Start with update:

\[ x_{k+1} = y_k - \alpha_k \nabla f(y_k) \]
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- Expand to input, output, state:

\[ \xi^{(1)}_{k+1} = (1 + \beta) \xi^{(1)}_k - \beta \xi^{(2)}_k - \alpha u_k \]
\[ \xi^{(2)}_{k+1} = \xi^{(1)}_k \]
\[ y_k = (1 + \beta) \xi^{(1)}_k - \beta \xi^{(2)}_k \]
\[ u_k = \nabla f(y_k) \]

- Block form:

\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
(1 + \beta)I_d & -\beta I_d & -\alpha I_d \\
I_d & 0_d & 0_d \\
\frac{(1 + \beta)}{d}I_d & -\beta I_d & 0_d
\end{bmatrix}
\]
Necessary conditions for convergence

- For convex problems, we need $u_\star = \nabla f(y_\star) = 0$
Necessary conditions for convergence

- For convex problems, we need \( u_\star = \nabla f(y_\star) = 0 \)
- Plug this into update rule: \( \xi_\star = A\xi_\star, \ y_\star = C\xi_\star \)
Quadratic case

Suppose \( f(y) = \frac{1}{2} y^T Q y - p^T y + r \) with \( ml_d \preceq Q \preceq Ll_d \).
Quadratic case

- Suppose $f(y) = \frac{1}{2} y^T Q y - p^T y + r$ with $ml_d \preceq Q \preceq Ll_d$.
- Then $\nabla f(y) = Qy - p = Q(y - y^*)$
Quadratic case

Suppose \( f(y) = \frac{1}{2} y^T Q y - p^T y + r \) with \( ml_d \preceq Q \preceq Ll_d \).

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\( y_k = C \xi_k \), so \( u_k = QC(\xi_k - \xi^*) \)
Quadratic case

- Suppose $f(y) = \frac{1}{2} y^T Q y - p^T y + r$ with $ml_d \preceq Q \preceq Ll_d$.
- Then $\nabla f(y) = Q y - p = Q(y - y_*)$
- $y_k = C\xi_k$, so $u_k = QC(\xi_k - \xi_*)$
- From state update: $\xi_{k+1} - \xi_* = (A + BQC)(\xi_k - \xi_*)$
Quadratic case

- Suppose \( f(y) = \frac{1}{2} y^T Q y - p^T y + r \) with \( ml_d \preceq Q \preceq Ll_d \).
- Then \( \nabla f(y) = Q y - p = Q (y - y*) \)
- \( y_k = C \xi_k \), so \( u_k = QC (\xi_k - \xi*) \)
- From state update: \( \xi_{k+1} - \xi* = (A + BQC)(\xi_k - \xi*) \)
- Hence the spectral radius \( \rho(T) \) of \( T := A + BQC \) determines convergence rate
Quadratic case

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- Then \( \nabla f(y) = Qy - p = Q(y - y_*) \)
- \( y_k = C\xi_k \), so \( u_k = QC(\xi_k - \xi_*) \)
- From state update: \( \xi_{k+1} - \xi_* = (A + BQC)(\xi_k - \xi_*) \)
- Hence the spectral radius \( \rho(T) \) of \( T := A + BQC \) determines convergence rate
- Using given properties of \( Q \), we can analytically tune the parameters and determine rate \( \rho \) for e.g. gradient descent
An alternative approach

Theorem

The spectral radius $\rho(T) < \rho$ if and only if there exists $P \succeq 0$ such that $T^TPT - \rho^2P \preceq 0$.

- If $\xi_{k+1} - \xi_* = T(\xi_k - \xi_*)$ then
  
  $$(\xi_{k+1} - \xi_*)^T P (\xi_{k+1} - \xi_*) < \rho^2(\xi_k - \xi_*)^T P (\xi_k - \xi_*)$$
An alternative approach

**Theorem**

The spectral radius $\rho(T) < \rho$ if and only if there exists $P \succeq 0$ such that $T^TPT - \rho^2P \preceq 0$.

- If $\xi_{k+1} - \xi_* = T(\xi_k - \xi_*)$ then
  \[
  (\xi_{k+1} - \xi_*)^T P(\xi_{k+1} - \xi_*) < \rho^2(\xi_k - \xi_*)^T P(\xi_k - \xi_*)
  \]
- Iterating this, if $\rho < 1$, then
  \[
  \|\xi_k - \xi_*\| < \sqrt{\text{cond}(P)} \rho^k \|\xi_0 - \xi_*\|
  \]
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Unknown nasty function

- Suppose \( u = \phi(y) \) (\( u \) and \( y \) are sequences and \( \phi \) is nasty)
Unknown nasty function

- Suppose $u = \phi(y)$ ($u$ and $y$ are sequences and $\phi$ is nasty)
  - $\phi$ is static and memoryless: $\phi(y_0, y_1, \ldots) = (g(y_0), g(y_1), \ldots)$
Suppose $u = \phi(y)$ ($u$ and $y$ are sequences and $\phi$ is nasty)

- $\phi$ is static and memoryless: $\phi(y_0, y_1, \ldots) = (g(y_0), g(y_1), \ldots)$
- Further, $g$ is $L$-Lipschitz: $\|g(y_1) - g(y_2)\| \leq L\|y_1 - y_2\|$
Suppose $u = \phi(y)$ ($u$ and $y$ are sequences and $\phi$ is nasty)

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If $u_\star = g(y_\star)$ then for any $k$,

$$
\begin{bmatrix}
  y_k - y_\star \\
  u_k - u_\star
\end{bmatrix}^T
\begin{bmatrix}
  L^2 I_d & 0_d \\
  0_d & -I_d
\end{bmatrix}
\begin{bmatrix}
  y_k - y_\star \\
  u_k - u_\star
\end{bmatrix} \geq 0
$$
Unknown nasty function

- Suppose \( u = \phi(y) \) (\( u \) and \( y \) are sequences and \( \phi \) is nasty)
  - \( \phi \) is static and memoryless: \( \phi(y_0, y_1, \ldots) = (g(y_0), g(y_1), \ldots) \)
  - Further, \( g \) is \( L \)-Lipschitz: \( \|g(y_1) - g(y_2)\| \leq L \|y_1 - y_2\| \)
- If \( u_* = g(y_*) \) then for any \( k \),
  \[
  \begin{bmatrix}
  y_k - y_* \\
  u_k - u_*
  \end{bmatrix}^T
  \begin{bmatrix}
  L^2 I_d & 0_d \\
  0_d & -I_d
  \end{bmatrix}
  \begin{bmatrix}
  y_k - y_* \\
  u_k - u_*
  \end{bmatrix} \geq 0
  \]
- This gives constraints on \( (y, u) \) – in fact, on each pair \( (y_k, u_k) \)
Suppose \( u = \phi(y) \) (\( u \) and \( y \) are sequences and \( \phi \) is nasty)

- \( \phi \) is static and memoryless: \( \phi(y_0, y_1, \ldots) = (g(y_0), g(y_1), \ldots) \)
- Further, \( g \) is \( L \)-Lipschitz: \( \|g(y_1) - g(y_2)\| \leq L\|y_1 - y_2\| \)

If \( u_\star = g(y_\star) \) then for any \( k \),

\[
\begin{bmatrix}
    y_k - y_\star \\
    u_k - u_\star
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\begin{bmatrix}
    L^2 I_d & 0_d \\
    0_d & -I_d
\end{bmatrix}
\begin{bmatrix}
    y_k - y_\star \\
    u_k - u_\star
\end{bmatrix} \geq 0
\]

This gives constraints on \((y, u)\) – in fact, on each pair \((y_k, u_k)\)

“Reference point” \((y_\star, u_\star)\) should make you think of \( \text{arg min} \)
Core idea

- Instead of analyzing a system containing $\phi$, throw away $\phi$ but keep the constraints on some auxiliary sequence $z = \Psi(y, u)$
Core idea

- Instead of analyzing a system containing $\phi$, throw away $\phi$ but keep the constraints on some auxiliary sequence $z = \Psi(y, u)$
- Any analysis that is valid for the constrained system is valid for the original
Modifying our dynamical system

- Auxiliary sequences $\zeta, z$ and map $\Psi$ so that $\zeta_0 = \zeta_*$,

$$
\zeta_{k+1} = A\psi \zeta_k + B^y_\psi y_k + B^u_\psi u_k \\
z_k = C\psi \zeta_k + D^y_\psi y_k + D^u_\psi u_k
$$
Modifying our dynamical system

- Auxiliary sequences $\zeta, z$ and map $\Psi$ so that $\zeta_0 = \zeta_*$,

$$
\zeta_{k+1} = A_{\Psi} \zeta_k + B_{\Psi}^y y_k + B_{\Psi}^u u_k \\
z_k = C_{\Psi} \zeta_k + D_{\Psi}^y y_k + D_{\Psi}^u u_k
$$

- If $\rho(A_{\Psi}) < 1$ then reference point $(\zeta_*, z_*)$ determined by $(y_*, u_*)$
Definition of IQCs

**Definition**

Let \( u = \phi(y) \) and \( z = \Psi(y, u) \). We say that \( \phi \) satisfies the
Definition of IQCs

Definition
Let \( u = \phi(y) \) and \( z = \Psi(y, u) \). We say that \( \phi \) satisfies the

- **Pointwise IQC** defined by \((\Psi, M, y_\star, u_\star)\) if for all sequences \( y \),

\[
(z_k - z_\star)^T M (z_k - z_\star) \geq 0 \quad \forall k
\]
Definition of IQCs

Definition

Let $u = \phi(y)$ and $z = \Psi(y, u)$. We say that $\phi$ satisfies the

- **Pointwise IQC** defined by $(\Psi, M, y_\star, u_\star)$ if for all sequences $y$,

  $$
  (z_k - z_\star)^T M (z_k - z_\star) \geq 0 \ \forall k
  $$

- **$\rho$-Hard IQC** defined by $(\Psi, M, \rho, y_\star, u_\star)$ if for all sequences $y$,

  $$
  \sum_{t=0}^{k} \rho^{-2t} (z_t - z_\star)^T M (z_t - z_\star) \geq 0 \ \forall k
  $$
Definition of IQCs

**Definition**

Let \( u = \phi(y) \) and \( z = \Psi(y, u) \). We say that \( \phi \) satisfies the

- **Pointwise IQC** defined by \((\Psi, M, y^*, u^*)\) if for all sequences \( y \),
  \[
  (z_k - z^*)^T M (z_k - z^*) \geq 0 \quad \forall k
  \]

- **\( \rho \)-Hard IQC** defined by \((\Psi, M, \rho, y^*, u^*)\) if for all sequences \( y \),
  \[
  \sum_{t=0}^{k} \rho^{-2t} (z_t - z^*)^T M (z_t - z^*) \geq 0 \quad \forall k
  \]

- **Hard IQC** if satisfies \( \rho \)-Hard IQC for \( \rho = 1 \)
Revisiting dynamical systems for first order methods

• Recall:

\[ \xi_{k+1} = A\xi_k + Bu_k \]
\[ y_k = C\xi_k \]
Revisiting dynamical systems for first order methods

- Recall:

\[ \xi_{k+1} = A\xi_k + Bu_k \]
\[ y_k = C\xi_k \]

- Combine with the map \( \Psi \) and eliminate \( y \):

\[
\begin{bmatrix}
\xi_{k+1} \\
\zeta_{k+1}
\end{bmatrix}
= \begin{bmatrix}
A & 0 \\
B^y \Psi C & A_{\Psi}
\end{bmatrix}
\begin{bmatrix}
\xi_k \\
\zeta_k
\end{bmatrix}
+ \begin{bmatrix}
B \\
B^u_{\Psi}
\end{bmatrix}
\begin{bmatrix}
u_k
\end{bmatrix}
\]

\[ z_k = \begin{bmatrix}
D^y_{\Psi} C & C_{\Psi}
\end{bmatrix}
\begin{bmatrix}
\xi_k \\
\zeta_k
\end{bmatrix}
+ D^u_{\Psi} u_k \]
Revisiting dynamical systems for first order methods

- Recall:

\[
\begin{align*}
\xi_{k+1} &= A\xi_k + Bu_k \\
y_k &= C\xi_k
\end{align*}
\]

- Combine with the map \( \Psi \) and eliminate \( y \):

\[
\begin{bmatrix}
\xi_{k+1} \\
\zeta_{k+1}
\end{bmatrix} =
\begin{bmatrix}
A & 0 \\
B_y C & A_\Psi
\end{bmatrix}
\begin{bmatrix}
\xi_k \\
\zeta_k
\end{bmatrix} +
\begin{bmatrix}
B \\
B_u \Psi
\end{bmatrix} u_k
\]

\[
z_k =
\begin{bmatrix}
D_y C & C_\Psi
\end{bmatrix}
\begin{bmatrix}
\xi_k \\
\zeta_k
\end{bmatrix} + D_u \Psi u_k
\]

- More succinctly:

\[
\begin{align*}
x_{k+1} &= \hat{A}x_k + \hat{B}u_k \\
z_k &= \hat{C}x_k + \hat{D}u_k
\end{align*}
\]
Main result

Theorem

Suppose \((\xi_*, \zeta_*, y_*, u_*, z_*)\) is a fixed point of the system. Suppose \(\phi\) satisfies the \(\rho\)-hard IQC defined by \((\Psi, M, \rho, y_*, u_*)\) for \(\rho \in [0, 1]\). If the LMI

\[
\begin{bmatrix}
\hat{A}^T P \hat{A} - \rho^2 P & \hat{A}^T P \hat{B} \\
\hat{B}^T P \hat{A} & \hat{B}^T P \hat{B}
\end{bmatrix} + \lambda \begin{bmatrix}
\hat{C} & \hat{D}
\end{bmatrix}^T M \begin{bmatrix}
\hat{C} & \hat{D}
\end{bmatrix} \preceq 0
\]

is feasible for some \(P \succ 0\) and \(\lambda \geq 0\), then for any \(\xi_0\) we have

\[
\|\xi_k - \xi_*\| \leq \sqrt{\text{cond}(P)} \rho^k \|\xi_0 - \xi_*\| \forall k.
\]
Main result

Theorem
Suppose \((\xi_*, \zeta_*, y_*, u_*, z_*)\) is a fixed point of the system. Suppose \(\phi\) satisfies the \(\rho\)-hard IQC defined by \((\Psi, M, \rho, y_*, u_*)\) for \(\rho \in [0, 1]\). If the LMI

\[
\begin{bmatrix}
\hat{A}^T P \hat{A} - \rho^2 P & \hat{A}^T P \hat{B} \\
\hat{B}^T P \hat{A} & \hat{B}^T P \hat{B}
\end{bmatrix} + \lambda \begin{bmatrix}
\hat{C} & \hat{D}
\end{bmatrix}^T M \begin{bmatrix}
\hat{C} & \hat{D}
\end{bmatrix} \preceq 0
\]

is feasible for some \(P \succ 0\) and \(\lambda \geq 0\), then for any \(\xi_0\) we have

\[
\|\xi_k - \xi_*\| \leq \sqrt{\text{cond}(P)} \rho^k \|\xi_0 - \xi_*\| \quad \forall k.
\]

Proof.
Multiply on both sides by \(\begin{bmatrix}(x_k - x_*)^T & (u_k - u_*)^T\end{bmatrix}\) and its transpose. Then use the definition of \(\rho\)-hard IQC to find that

\[
\|x_k - x_*\| \leq \sqrt{\text{cond}(P)} \rho^k \|x_0 - x_*\|.
\]

Finally, use \(\zeta_0 = \zeta_*\), \(x = (\xi, \zeta)\), and the triangle inequality. \(\square\)
A few notes

- Pointwise IQC satisfied $\implies \rho$-hard IQC satisfied for any $\rho$, so find the smallest $\rho$ with the LMI feasible.
A few notes

- Pointwise IQC satisfied \( \Rightarrow \) \( \rho \)-hard IQC satisfied for any \( \rho \), so find the smallest \( \rho \) with the LMI feasible

- Hard IQC means 1-hard IQC, which implies bounded iterates but not convergence
A few notes

- Pointwise IQC satisfied $\Rightarrow \rho$-hard IQC satisfied for any $\rho$, so find the smallest $\rho$ with the LMI feasible.
- Hard IQC means 1-hard IQC, which implies bounded iterates but not convergence.
- If $\phi$ satisfies multiple IQCs, replace $\lambda M$ with a block diagonal matrix with $\lambda_i M_i$ on the diagonal.
Lemma (Sector IQC)

Suppose \( f_k \in S(m, L) \) and \( u_\star = \nabla f_k(y_\star) \) for all \( k \). Let \( \phi = (\nabla f_0, \nabla f_1, \ldots) \). If \( u = \phi(y) \), then \( \phi \) satisfies the pointwise IQC defined by

\[
\begin{bmatrix}
L I_d & -I_d \\
-m I_d & I_d
\end{bmatrix}
\quad \text{and} \quad
M =
\begin{bmatrix}
0_d & I_d \\
I_d & 0_d
\end{bmatrix}.
\]

This corresponds to the constraint that for all sequences \( y \),

\[
\begin{bmatrix}
y_k - y_\star \\
u_k - u_\star
\end{bmatrix}^T
\begin{bmatrix}
-2m L I_d & (L + m) I_d \\
(L + m) I_d & -2 I_d
\end{bmatrix}
\begin{bmatrix}
y_k - y_\star \\
u_k - u_\star
\end{bmatrix} \geq 0.
\]
Lemma (Sector IQC)

Suppose \( f_k \in S(m, L) \) and \( u_\star = \nabla f_k(y_\star) \) for all \( k \). Let \( \phi = (\nabla f_0, \nabla f_1, \ldots) \).

If \( u = \phi(y) \), then \( \phi \) satisfies the pointwise IQC defined by

\[
\psi = \begin{bmatrix}
L_l I_d & -I_d \\
-m I_d & I_d
\end{bmatrix}
\text{ and } M = \begin{bmatrix}
0_d & I_d \\
I_d & 0_d
\end{bmatrix}.
\]

This corresponds to the constraint that for all sequences \( y \),

\[
\begin{bmatrix}
y_k - y_\star \\
u_k - u_\star
\end{bmatrix}^T
\begin{bmatrix}
-2m L_l I_d & (L + m) I_d \\
(L + m) I_d & -2 I_d
\end{bmatrix}
\begin{bmatrix}
y_k - y_\star \\
u_k - u_\star
\end{bmatrix} \geq 0.
\]

Note: this \( \psi \) corresponds to no \( \zeta \), and

\[
z = \psi \begin{bmatrix}
y \\
u
\end{bmatrix} = \begin{bmatrix}
Ly - u \\
-my + u
\end{bmatrix}.
\]
Proof.

If $f$ has $L$-Lipschitz gradient, then we have

$$(x_1 - x_2)^T(\nabla f(x_1) - \nabla f(x_2)) \geq \frac{1}{L} \|\nabla f(x_1) - \nabla f(x_2)\|^2$$

which is known as co-coercivity. Note $f(x) - \frac{m}{2}\|x\|^2 \in S(0, L - m)$ has Lipschitz gradient with parameter $L - m$. By co-coercivity, and replacing $x_1, x_2$ with $y_k, y_\star$, etc., we see that

$$(m + L)(y_k - y_\star)^T(u_k - u_\star) \geq mL\|y_k - y_\star\|^2 + \|u_k - u_\star\|^2$$

which we can rearrange into matrix form.
Lemma (IQC for general convex functions)

Suppose $f_k \in S(0, \infty)$ and $u_\star \in \partial f_k(y_\star)$ for all $k$. Let $\phi$ be such that $u_k \in \partial f_k(y_k)$ for all $k$. Then $\phi$ satisfies the pointwise IQC defined by

$$
\Psi = \begin{bmatrix} I_d & 0 \\ 0 & I_d \end{bmatrix} = I_{2d} \quad \text{and} \quad M = \begin{bmatrix} 0_d & I_d \\ I_d & 0_d \end{bmatrix}.
$$

This corresponds to the constraint that for all sequences $y$,

$$
\begin{bmatrix} y_k - y_\star \\ u_k - u_\star \end{bmatrix}^T \begin{bmatrix} 0_d & I_d \\ I_d & 0_d \end{bmatrix} \begin{bmatrix} y_k - y_\star \\ u_k - u_\star \end{bmatrix} \geq 0.
$$
Lemma (IQC for general convex functions)

Suppose \( f_k \in S(0, \infty) \) and \( u_* \in \partial f_k(y_*) \) for all \( k \). Let \( \phi \) be such that \( u_k \in \partial f_k(y_k) \) for all \( k \). Then \( \phi \) satisfies the pointwise IQC defined by

\[
\psi = \begin{bmatrix} I_d & 0 \\ 0 & I_d \end{bmatrix} = I_{2d} \quad \text{and} \quad M = \begin{bmatrix} 0_d & I_d \\ I_d & 0_d \end{bmatrix}.
\]

This corresponds to the constraint that for all sequences \( y \),

\[
\begin{bmatrix} y_k - y_* \\ u_k - u_* \end{bmatrix}^T \begin{bmatrix} 0_d & I_d \\ I_d & 0_d \end{bmatrix} \begin{bmatrix} y_k - y_* \\ u_k - u_* \end{bmatrix} \geq 0.
\]

Proof.

This is equivalent to \( (y_k - y_*)^T(u_k - u_*) \geq 0 \), i.e. that the subdifferential of a convex function is a monotone operator. (combine \( f(y_*) \geq f(y_k) + u_k^T(y_* - y_k) \) and vice-versa per EE236C)
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4 Case studies (a.k.a. actually applying IQCs)
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5 Dealing with noise
6 Conclusion
SDP tractibility

- We prove convergence by finding $P \succ 0 \ldots$ how big is $P$?
SDP tractibility

- We prove convergence by finding $P > 0$ . . . how big is $P$?
- Our LMI has the term $\hat{A}^T P \hat{A}$, where $\hat{A}$ operates on $(\xi, \zeta)$. Hence $P$ is $n \times n$ where $(\xi, \zeta) \in \mathbb{R}^n$. Better than Drori and Teboulle '13, where the SDP scales with the number of time steps, but still too large to e.g. analyze gradient descent in high dimensions.
SDP tractibility

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- Our LMI has the term $\hat{A}^T P \hat{A}$, where $\hat{A}$ operates on $(\xi, \zeta)$. Hence $P$ is $n \times n$ where $(\xi, \zeta) \in \mathbb{R}^n$.
- Better than Drori and Teboulle ’13, where the SDP scales with the number of time steps, but still too large to e.g. analyze gradient descent in high dimensions.
Structure in our linear maps

- First-order methods in dynamical system form often have block-diagonal structure

$$
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix} =
\begin{bmatrix}
(1 + \beta) I - \beta I & 0 \\
0 & (1 + \beta) I - \beta I
\end{bmatrix}
$$

For example,

$$
A =
\begin{bmatrix}
1 + \beta & -\beta \\
1 & 0
\end{bmatrix} \otimes I_d
$$

Even our IQCs have this form, e.g.

$$
\Psi =
\begin{bmatrix}
L & -1 \\
-\bar{m} & 0
\end{bmatrix} \otimes I_d
$$

and

$$
M =
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix} \otimes I_d
$$

for the sector IQC.
Structure in our linear maps

- First-order methods in dynamical system form often have block-diagonal structure
- Nesterov’s accelerated gradient method has

\[ \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} (1 + \beta)I_d & -\beta I_d & -\alpha I_d \\ I_d & 0_d & 0_d \\ (1 + \beta)I_d & -\beta I_d & 0_d \end{bmatrix} \end{bmatrix} \]
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for the sector IQC.
Making the SDP small

- If each matrix \( (\hat{A}, \hat{B}, \hat{C}, \hat{D}, M) \) from the LMI has the form e.g. \( \hat{A} = \hat{A}_0 \otimes I_d \) then we can instead solve the smaller LMI (which is the equivalent of the \( d = 1 \) case):

\[
\begin{bmatrix}
\hat{A}_0^T P_0 \hat{A}_0 - \rho^2 P_0 & \hat{A}_0^T P_0 \hat{B}_0 \\
\hat{B}_0^T P_0 \hat{A}_0 & \hat{B}_0^T P_0 \hat{B}_0
\end{bmatrix} + \lambda \begin{bmatrix} \hat{C}_0 & \hat{D}_0 \end{bmatrix}^T M_0 \begin{bmatrix} \hat{C}_0 & \hat{D}_0 \end{bmatrix} \preceq 0
\]

We can get feasible \( P_0 \) from \( P \) and vice-versa, so solving this smaller SDP is completely equivalent.

In the first order methods we have looked at so far, this means \( P_0 \) is no bigger than \( 2 \times 2 \).
Making the SDP small

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- In the first order methods we have looked at so far, this means \(P_0\) is no bigger than \(2 \times 2\)
Using the sector IQC and the dimensionality reduction, the LMI for
gradient descent is

\[
\begin{bmatrix}
(1 - \rho^2)P & -\alpha P \\
-\alpha P & \alpha^2 P
\end{bmatrix} + \lambda \begin{bmatrix}
-2mL & L + m \\
L + m & -2
\end{bmatrix} \preceq 0
\]

For \(\alpha = 2L + m\) (optimal for \(f\) quadratic), we find
\[
\lambda \geq \frac{2(L + m)^2}{2mL}
\]
and
\[
\rho^2 \geq \frac{1}{2} \lambda \left(\frac{1}{2}L - m\right)^2
\]
which yields optimal \(\rho = L - m\).

Can reformulate LMI so that it is linear in \((\rho^2, \lambda, \alpha)\). Hence, can
answer "what range of stepsizes yield a given rate?" etc.
Analytic results for gradient descent

- Using the sector IQC and the dimensionality reduction, the LMI for gradient descent is
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  \end{bmatrix} \preceq 0
  \]

- For \( \alpha = \frac{2}{L + m} \) (optimal for \( f \) quadratic), we find \( \lambda \geq \frac{2}{(L + m)^2} \) and \( \rho^2 \geq \frac{1}{2} \lambda (L - m)^2 \) which yields optimal \( \rho = \frac{L - m}{L + m} \).
Analytic results for gradient descent

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Analyzing Nesterov’s method

- Analyze \( \alpha = 1/L \) and \( \beta = (\sqrt{L} - \sqrt{m}) / (\sqrt{L} + \sqrt{m}) \) which are optimal when \( f \) is quadratic

Solve the LMI numerically. LMI is no longer linear in \( \rho \) but can find optimal via bisection search.

Sector IQC actually fails for high \( \kappa = L/m \), but more sophisticated weighted off-by-one IQC works.
Analyzing Nesterov’s method

- Analyze $\alpha = 1/L$ and $\beta = (\sqrt{L} - \sqrt{m})/(\sqrt{L} + \sqrt{m})$ which are optimal when $f$ is quadratic.
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Analyzing Nesterov’s method

- Analyze $\alpha = 1/L$ and $\beta = (\sqrt{L} - \sqrt{m})/(\sqrt{L} + \sqrt{m})$ which are optimal when $f$ is quadratic
- Solve the LMI numerically. LMI is no longer linear in $\rho^2$ but can find optimal via bisection search
- Sector IQC actually fails for high $\kappa = L/m$, but more sophisticated weighted off-by-one IQC works
Convergence rate v. condition ratio

![Graph showing convergence rate vs. condition ratio]

- LMI (sector)
- LMI (weighted off-by-one)
- Optimal Gradient rate
- Standard Nesterov rate
- Theoretical lower bound

Convergence rate $\rho$ vs. Condition ratio $L/m$
Robustness of Nesterov’s method

- Sector IQC (which allows different functions \( f_k \) for each \( k \)) fails for large \( \kappa \), unlike gradient descent

Optimal parameters \( \alpha = \frac{4}{3L + m} \) and \( \beta = \sqrt{3 \kappa + 1} - 2 \sqrt{3 \kappa + 1} + 2 \) cause sector IQC to fail faster
Robustness of Nesterov’s method

- Sector IQC (which allows different functions $f_k$ for each $k$) fails for large $\kappa$, unlike gradient descent.
- Optimal parameters $\alpha = 4/(3L + m)$ and $\beta = \frac{\sqrt{3\kappa+1} - 2}{\sqrt{3\kappa+1} + 2}$ cause sector IQC to fail faster.
Robustness of Nesterov’s method

- Sector IQC (which allows different functions $f_k$ for each $k$) fails for large $\kappa$, unlike gradient descent
- Optimal parameters $\alpha = 4/(3L + m)$ and $\beta = \frac{\sqrt{3\kappa+1} - 2}{\sqrt{3\kappa+1} + 2}$ cause sector IQC to fail faster
- In some sense, gradient descent, and even the suboptimal parameters $\alpha, \beta$ more robust than fully optimal Nesterov
Robustness of heavy ball method

- Recall the heavy ball method:

\[ x_{k+1} = x_k - \alpha \nabla f(x_k) + \beta (x_k - x_{k-1}) \]
Robustness of heavy ball method

- Recall the heavy ball method:

\[ x_{k+1} = x_k - \alpha \nabla f(x_k) + \beta (x_k - x_{k-1}) \]

- For quadratic-optimal \( \alpha, \beta \) for heavy ball method, not even weighted off-by-one IQC can guarantee convergence for \( \kappa = L/m \) at least \( \approx 18 \).
Robustness of heavy ball method

- Recall the heavy ball method:
  \[ x_{k+1} = x_k - \alpha \nabla f(x_k) + \beta (x_k - x_{k-1}) \]

- For quadratic-optimal \( \alpha, \beta \) for heavy ball method, not even weighted off-by-one IQC can guarantee convergence for \( \kappa = \frac{L}{m} \) at least \( \approx 18 \).

- Informs a function \( f(x) \) with piecewise-linear gradient and \( \kappa = \frac{L}{m} = 25 \) for which heavy ball method optimized for quadratics does not converge.
ADMM background

- ADMM seeks to solve the problem

\[
\begin{align*}
\text{minimize} & \quad f(x) + g(z) \\
\text{subject to} & \quad Ax + Bz = c
\end{align*}
\]
ADMM background

ADMM seeks to solve the problem

$$\text{minimize} \quad f(x) + g(z)$$
$$\text{subject to} \quad Ax + Bz = c$$

Updates are of the form:

$$x_{k+1} = \arg \min_x f(x) + \frac{\rho}{2} \|Ax + Bz_k - c + u_k\|^2$$
$$z_{k+1} = \arg \min_z g(z) + \frac{\rho}{2} \|Ax_{k+1} + Bz - c + u_k\|^2$$
$$u_{k+1} = u_k + Ax_{k+1} + Bz_{k+1} - c$$
ADMM background

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x_{k+1} &= \arg \min_x f(x) + \frac{\rho}{2} \|Ax + Bz_k - c + u_k\|^2 \\
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u_{k+1} &= u_k + Ax_{k+1} + Bz_{k+1} - c
\end{align*}
\]

- Over-relaxed ADMM given by replacing \(Ax_{k+1}\) with \(\alpha Ax_{k+1} - (1 - \alpha)(Bz_k - c)\) in \(z\) and \(u\) updates. Typically \(\alpha \in (0, 2]\)
ADMM as a dynamical system

- Assume \( f \in S(m, L) \) and \( g \in S(0, \infty) \). Then instead of one sequence \( u_k \) of gradients of \( y_k \), instead have two sequences \( \beta_k = \nabla \hat{f}(r_k) \) and \( \gamma_k \in \partial \hat{g}(s_k) \) (\( \hat{f} \) and \( \hat{g} \) are versions of \( f, g \) scaled by \( A, B, \rho \))
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- Then we can write $x, z$ iterates (now called $r, s$) in terms of $\beta, \gamma$, e.g.

$$x_{k+1} = A^{-1} \arg\min_r f(A^{-1}r) + \frac{\rho}{2} \|r + s_k - c + u_k\|^2$$
ADMM as a dynamical system

- Assume $f \in S(m, L)$ and $g \in S(0, \infty)$. Then instead of one sequence $u_k$ of gradients of $y_k$, instead have two sequences $\beta_k = \nabla \hat{f}(r_k)$ and $\gamma_k \in \partial \hat{g}(s_k)$ ($\hat{f}$ and $\hat{g}$ are versions of $f, g$ scaled by $A, B, \rho$).

- Then we can write $x, z$ iterates (now called $r, s$) in terms of $\beta, \gamma$, e.g.

\[
x_{k+1} = A^{-1} \arg \min_r f(A^{-1}r) + \frac{\rho}{2} \|r + s_k - c + u_k\|^2
\]

\[\Rightarrow r_{k+1} = \arg \min_r \hat{f}(r) + \frac{1}{2} \|r + s_k - c + u_k\|^2\]
ADMM as a dynamical system

- Assume \( f \in S(m, L) \) and \( g \in S(0, \infty) \). Then instead of one sequence \( u_k \) of gradients of \( y_k \), instead have two sequences \( \beta_k = \nabla \hat{f}(r_k) \) and \( \gamma_k \in \partial \hat{g}(s_k) \) (\( \hat{f} \) and \( \hat{g} \) are versions of \( f, g \) scaled by \( A, B, \rho \))
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\[
\begin{align*}
    x_{k+1} &= A^{-1} \arg\min_r f(A^{-1}r) + \frac{\rho}{2} \| r + s_k - c + u_k \|^2 \\
    \Rightarrow \quad r_{k+1} &= \arg\min_r \hat{f}(r) + \frac{1}{2} \| r + s_k - c + u_k \|^2
\end{align*}
\]

and via optimality conditions implies

\[
r_{k+1} = -s_k - u_k + c - \beta_{k+1}.
\]
IQC for ADMM

- One IQC for each of $f, g$
IQC for each of $f, g$

- Sector IQC for $f \in S(m, L)$ and corresponding iterates

Put $M_1$ and $M_2$ into a block diagonal matrix and solve

Given fixed $\alpha, \rho, m, L$, can bisection search on convergence rates.
• One IQC for each of \( f, g \)
• Sector IQC for \( f \in S(m, L) \) and corresponding iterates
• More general pointwise IQC for \( g \in S(0, \infty) \) and corresponding iterates
IQC for ADMM

- One IQC for each of $f, g$
- Sector IQC for $f \in S(m, L)$ and corresponding iterates
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- Put $M_1$ and $M_2$ into a block diagonal matrix and solve
One IQC for each of $f, g$
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More general pointwise IQC for $g \in S(0, \infty)$ and corresponding iterates
Put $M_1$ and $M_2$ into a block diagonal matrix and solve
Given fixed $\alpha, \rho, m, L$, can bisection search on convergence rates $\tau$. 
Some results for ADMM

- Prior work limits us to $\alpha \in (0, 2)$ but depending on $\kappa$, we can find convergent $\alpha$ larger than 2.
Some results for ADMM

- Prior work limits us to $\alpha \in (0, 2)$ but depending on $\kappa$, we can find convergent $\alpha$ larger than 2
- Also able to analytically construct certificates $\lambda, P$ that work for large enough $\kappa$ (for $\alpha \in (0, 2)$ and specific choice of $\rho$
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Multiplicative gradient noise

- Suppose instead of observing $\nabla f(y)$, we see $u_k = \nabla f(y_k) + r_k$, where $\|r_k\| \leq \delta \|\nabla f(y_k)\|$
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If $w_k$ is true gradient, we observe $u_k$ with $\|u_k - w_k\| \leq \delta \|w_k\|$
Multiplicative gradient noise

- Suppose instead of observing $\nabla f(y)$, we see $u_k = \nabla f(y_k) + r_k$, where $\|r_k\| \leq \delta \|\nabla f(y_k)\|$.
- If $w_k$ is true gradient, we observe $u_k$ with $\|u_k - w_k\| \leq \delta \|w_k\|$.
- In IQC form:

$$\begin{bmatrix} w_k \\ u_k \end{bmatrix}^T \begin{bmatrix} \delta^2 - 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w_k \\ u_k \end{bmatrix} \geq 0$$
Multiplicative gradient noise

- Suppose instead of observing $\nabla f(y)$, we see $u_k = \nabla f(y_k) + r_k$, where $\|r_k\| \leq \delta \|\nabla f(y_k)\|$
- If $w_k$ is true gradient, we observe $u_k$ with $\|u_k - w_k\| \leq \delta \|w_k\|$
- In IQC form:
  $$\begin{bmatrix} w_k \\ u_k \end{bmatrix}^T \begin{bmatrix} \delta^2 - 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} w_k \\ u_k \end{bmatrix} \geq 0$$
- We can use nearly the same LMI after augmenting our state with $w$, i.e. we keep track of $(y, u, w)$, and instead solve for $3 \times 3 \ P$ for e.g. Nesterov’s method
Nesterov’s method convergence rates with noisy gradient

![Graph showing convergence rates with noisy gradient](image)

- Convergence rate $\rho$
- Condition ratio $L/m$
- Rates for different values of $\delta$: $\delta \in \{0.05, 0.1, 0.2, 0.3, 0.4, 0.5\}$
- Standard Nesterov rate
Summary

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- IQCs exist which capture standard properties of convex functions
Many optimization methods are (almost) linear dynamical systems

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- IQCs exist which capture standard properties of convex functions
- Automatic numerical convergence rate bounds whenever we have bounds on $m, L$ and (in the noisy case) $\delta$
Summary

- Many optimization methods are (almost) linear dynamical systems
- IQCs can replace nonlinearities in these systems
- IQCs exist which capture standard properties of convex functions
- Automatic numerical convergence rate bounds whenever we have bounds on $m, L$ and (in the noisy case) $\delta$
- Hence easy parameter tuning/algorithm design
Future work

- Analytic proofs doable if we can solve small SDPs in closed form
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- We don’t usually know $m, L$; connections to e.g. adaptive control?
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- We don’t usually know $m, L$; connections to e.g. adaptive control?
- More sophisticated parameter search needed if we want more steps of memory

Noise analysis is far from complete; IQCs that are valid in expectation?
We translated convexity properties into IQCs; are there useful IQCs for certain nonconvex functions?
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